Math 108, Fall 2013

HW 8 SOLUTIONS

5.1 2(k) inf. 2(l) inf. 2(m) equals $\{2\}$, thus finite (observe $x^2+x = x(x+1)$)

2(o) finite (as $x \le 10$)

4 The inverse of f is given by $f^{-1}(c,d) = (h^{-1}(c), g^{-1}(d))$.

6(b) Contradiction: If B is finite, and $A \subseteq B$, then A is finite. But we are assuming A is infinite. □

11(d) If $A = B \neq \emptyset$, $\overline{\overline{A \cup B}} = \overline{\overline{A}}$, but $\overline{\overline{A}} + \overline{\overline{B}} = 2\overline{\overline{A}} > \overline{\overline{A}}$.

12 Contradiction: If $B-A$ is finite, then $B = A \cup (B-A)$ is finite.

21(b) There are $2^{10} = 1024$ subsets of S, with sums between 0 and 1000. Thus there are two subsets A and B of S with equal sums, and then $A-B$ and $B-A$ are disjoint subsets with equal sums. □

22(a) C. The result is true, but unless $A \cap B = \emptyset$, it is not true that $A \cup B \approx \mathbb{N}_{m+n}$. The proof is wrong.

5.2 3(c) $f : \mathbb{Z} \to 3\mathbb{Z}$ given by $f(x) = 3x$ is a bijection. Thus $3\mathbb{Z} \approx \mathbb{Z} \approx \mathbb{N}$. (Or use the fact that $3\mathbb{Z} \subseteq \mathbb{Z} \approx \mathbb{N}$.)

4(c) We prove that $(0,1) \approx (-\infty, b)$ by using the bijection $f : (0,1) \xrightarrow{\text{onto}} (-\infty, b)$, given by $f(x) = -\frac{1}{x} + 1 + b$. (Another possibility is $f(x) = \ln x + b$.)

7(b) c, 7(d) $\aleph_0$, 7(g) $\aleph_0$ (as $\mathbb{Z} \times \mathbb{Z} \approx \mathbb{N} \times \mathbb{N} \approx \mathbb{N}$)

12(b) F. The claim is true, but the proof is wrong: if A is infinite, it does not follow that $A \approx \mathbb{N}$.

5.3 2 The set is subset of $\mathbb{Q}$, and is infinite as it contains $\{\frac{1}{3}2^n : n \in \mathbb{N}\}$.

9(c) $\mathbb{Q} \cap (1,2) \subseteq \mathbb{Q}$, therefore it is countable. It is also infinite, as it contains the infinite set $\{1 + \frac{1}{n} : n \in \mathbb{N}, n \ge 2\}$.

9(e) The set is a countable union of countable sets.

9(f) The set is a subset of $\mathbb{Q}$, and infinite, as it contains the infinite set $\{\frac{1}{2^k} : k \in \mathbb{N}\}$.

10(a) No. A may be finite. 10(b) No. $\mathbb{N} \subseteq \mathbb{R}$, but $\mathbb{R}$ is not denumerable.

10(d) Yes. $\mathbb{Q} - \mathbb{Z} \subseteq \mathbb{Q}$, and $\mathbb{Q} - \mathbb{Z}$ is infinite, as it contains $\{\frac{1}{n} : n \in \mathbb{N}, n \ge 2\}$

13(a) Assume $f : \mathbb{N} \to S$ is onto. Let $f(n)_i$ be the i-th element of the sequence $f(n)$. Define a sequence b by $b_i = \begin{cases} 1 & \text{if } f(i)_i = 0 \\ 0 & \text{if } f(i)_i = 1 \end{cases}$. Then $b \neq f(n)$ for every n, so f is not onto.

HW 8 SOLUTIONS, CONTINUED

14(a) Let S be the set. The function $f: A \to S$ given by $f(x) = \{x\}$ is a bijection. Thus $S \approx A$. □

14(b) Let S be the set, and assume $f: A \xrightarrow{\approx} \mathbb{N}$ is a bijection. Let T be the set of all 2-element subsets of $\mathbb{N}$. Then $g: S \to T$ given by $g(B) = f(B)$ (note: $f(B)$ is the image of B under f) is a bijection. To show that T is denumerable, define $h: T \xrightarrow{\approx} \{(x,y) : x, y \in \mathbb{N}, x<y\}$ by $h(C) =$ (smallest elt. of C, largest elt. of C), for every pair $C \subseteq \mathbb{N}$. Thus $S \approx T \approx \{(x,y) : x, y \in \mathbb{N}, x<y\} \subseteq \mathbb{N} \times \mathbb{N}$, so S is countable, but S is clearly infinite, as it includes infinite set $\{\{n, n+1\} : n \in \mathbb{N}\}$. □

$\boxed{5.4}$ 2. $\overline{\overline{\mathcal{P}(\mathbb{N})}} = c = \overline{\overline{\mathbb{R}}} < \overline{\overline{\mathcal{P}(\mathbb{R})}}$.

5(a). No. Take $A = \mathbb{Z}$, and $B = \mathbb{N}$: $\overline{\overline{A}} = \overline{\overline{B}}$ but $A \not\subseteq B$.

5(e). No. Now take $A = \mathbb{N}$, and $B = \mathbb{Z}$; $A \cup B = B$ and $\overline{\overline{A}} = \overline{\overline{B}}$.

8(a) Omitted.

8(b) $\overline{\overline{\{0,5\}}} = 2$, $\overline{\overline{[0,5]}} = c$, $\overline{\overline{\{0,3,5\}}} = 3$, $\overline{\overline{\mathbb{R} - \{3\}}} = c$,
$\overline{\overline{\mathcal{P}(\{0,5\})}} = 4$, $\overline{\overline{\mathcal{P}((0,5))}} = \overline{\overline{\mathcal{P}(\mathbb{R})}}$, $\overline{\overline{(0,5) - \{3\}}} = c$, $\overline{\overline{\mathbb{R} - \mathbb{N}}} = c$.