
Discussion problems 1

Note. These are 25/125A review problems, mostly collected from past exams.

1. Suppose that \( f : [0, 1] \to \mathbb{R} \) is continuous, and \( f(x) > 0 \) for all \( x \in [0, 1] \). (a) Show that the function \( g : [0, 1] \to \mathbb{R} \) given by \( g(x) = 1/f(x) \) is bounded. (b) Does the result from (a) remain true if the closed interval \([0, 1]\) is replaced by the open interval \((0, 1)\)?

2. Suppose that \( f, g : \mathbb{R} \to \mathbb{R} \) are uniformly continuous functions. (a) Must their product \( fg \) be uniformly continuous? (b) Answer the question in (a) if both \( f \) and \( g \) are uniformly continuous and bounded.

3. Call \( f : (0, \infty) \to (0, \infty) \) slow if \( f \) is nondecreasing and \( \lim_{x \to \infty} f(2x)/f(x) = 1 \). (a) Give an example of a slow function. (b) Is \( f \) given by \( f(x) = e^{\sqrt{\log x}} \) slow? (c) Show that if \( f \) is slow, then \( \lim_{x \to \infty} f(ax)/f(x) = 1 \) for every \( a > 0 \). (d) Show that if \( f \) is slow, then \( \lim_{x \to \infty} f(x)/x^a = 0 \) for every \( a > 0 \). (e) Show that the reverse implication from (d) does not hold.

4. Assume \( k \geq 0 \) and that \( f : \mathbb{R} \to \mathbb{R} \) is such its \( k \)th derivative \( f^{(k)} \) exists and is continuous everywhere. Assume that there exists \( k + 1 \) different numbers \( x_1, \ldots, x_{k+1} \) so that \( f(x_1) = \cdots = f(x_{k+1}) = 0 \). Show that \( f^{(k)}(z) = 0 \) for some \( z \in \mathbb{R} \). Give a careful inductive proof.

5. Show that the functions (a) \( f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \) and (b) \( f(x) = \sum_{n=1}^{\infty} \frac{\log(1+nx^2)}{n^2} \) are defined and continuous everywhere. Does the series in (b) converge uniformly on \( \mathbb{R} \)?

6. Let \( f(x) = 1 + \sum_{n=1}^{\infty} a_n x^{3n} \), where

\[
a_n = \frac{1 \cdot 4 \cdot 7 \cdots (3n - 2)}{(3n)!}.
\]

(a) Prove that the series converges for every \( x \in \mathbb{R} \). (b) Show that \( f''(x) = xf(x) \) for every \( x \in \mathbb{R} \).

7. Let \( S \) be the set of all binary sequences \( \{a = (a_1a_2, \ldots) : a_i \in \{0, 1\} \text{ for all } i\} \). For \( a = (a_1a_2, \ldots) \in S \) and \( b = (\beta_1\beta_2, \ldots) \in S \), \( a \neq b \), let \( N(a, b) \) be the smallest \( i \in \mathbb{N} \) so that \( a_i \neq \beta_i \). Define \( d : S \times S \to \mathbb{R} \) by setting

\[
d(a, b) = \begin{cases} 
2^{-N(a, b)} & \text{if } a \neq b \\
0 & \text{otherwise}
\end{cases}
\]

(a) Prove that \( d \) is a metric on \( S \). (b) Let \( z = (000\ldots) \) be the zero sequence. Give an example of a sequence \( a_n \in S \), \( a_n \neq z \) for all \( n \), so that \( \lim a_n = z \).
Brief solutions

1. (a) Let \( m = \inf \{ f(x) : x \in [0, 1] \} \). As \( f \) is continuous on a compact interval, it attains its minimum at an \( x_0 \in [0, 1] \). Thus \( m = f(x_0) > 0 \). It follows that \( m \leq f(x) \) and \( 0 < 1/f(x) \leq 1/m \) for every \( x \in [0, 1] \).

(b) No. Take \( f(x) = x \) as a counterexample.

2. (a) No. Take \( f(x) = g(x) = x \). The product is \( h(x) = x^2 \) which is not uniformly continuous: for \( \delta > 0 \), \( h(n+\delta) - h(n) \geq 2n\delta \geq 2 \) if \( n \geq 1/\delta \).

(b) Yes. By the assumption, there exists an \( M \in \mathbb{R} \) so that \( f(x) \leq M \) and \( g(x) \leq M \) for all \( x \in \mathbb{R} \). Pick an \( \epsilon > 0 \). Then pick a \( \delta > 0 \) so that \( |x_1 - x_2| < \delta \) implies \( |f(x_2) - f(x_1)| < \epsilon/(2M) \) and \( |g(x_2) - g(x_1)| < \epsilon/(2M) \), which can be done as \( f \) and \( g \) are uniformly continuous. Then, if \( |x_1 - x_2| < \delta \),

\[
|f(x_2)g(x_2) - f(x_1)g(x_1)| = |f(x_2)g(x_2) - f(x_1)g(x_2) + f(x_1)g(x_2) - f(x_1)g(x_1)| \\
\leq |f(x_2) - f(x_1)||g(x_2)| + |f(x_1)||g(x_2) - g(x_1)| \\
\leq M|f(x_2) - f(x_1)| + M|g(x_2) - g(x_1)| < \epsilon.
\]

3. (a) The simplest example is \( f(x) = \log(x+1) \).

(b) Yes, as

\[
\sqrt{\log(2x)} = \sqrt{\log x} + \frac{\log 2}{\sqrt{\log(2x)} + \sqrt{\log x}}
\]

and the last summand goes to 0.

(c) We may assume that \( a \geq 1 \): if \( a < 1 \) we flip the fraction and make the substitution \( ax = t \). Let us first assume \( a = 2^k \) for some \( k > 1 \). Then

\[
\frac{f(ax)}{f(x)} = \frac{f(2x)}{f(x)} \frac{f(2^2x)}{f(2x)} \cdots \frac{f(2^kx)}{f(2^{k-1}x)} \to 1
\]

as every one of the \( k \) factors converges to 1. Now for any \( a > 1 \) there exists a \( k \) so that \( a \leq 2^k \), and then by monotonicity

\[
1 \leq \frac{f(ax)}{f(x)} \leq \frac{f(2^kx)}{f(x)}
\]

and the result follows by the sandwich theorem.

(d) Fix an \( a > 0 \) and pick an \( \epsilon > 0 \) so that \( 1 + \epsilon < 2^{a/2} \). As \( f \) is slow, then there is an \( x_0 \) so that, for \( x \geq x_0 \), \( f(2x) \leq f(x)(1 + \epsilon) \). Therefore, for each integer \( k \geq 0 \)

\[
f(2^kx_0) \leq (1 + \epsilon)f(2^{k-1}x_0) \leq (1 + \epsilon)^2f(2^{k-2}x_0) \leq \cdots \leq (1 + \epsilon)^k f(x_0) \leq 2^{ak/2} f(x_0).
\]

Take an \( x \geq x_0 \). Then there exists a an integer \( k \geq 1 \) so that \( 2^{k-1}x_0 \leq x < 2^kx_0 \) and then by monotonicity

\[
f(x) \leq f(2^kx_0) \leq (2^k)_{a/2} f(x_0) \leq (2^{k-1}x_0)^{a/2} f(x_0) = 2^{a/2}x_0^{-a/2} f(x_0) \cdot x^{a/2}.
\]

Clearly, divided by \( x^{a/2} \), the last expression goes to 0 as \( x \to \infty \).

(e) Let \( x_n = e^{n^2} \) for \( n \geq 0 \). Then let \( f(x) = e^x \) for \( x \in (x_n, x_{n+1}] \). Also let \( f(x) = 1 \) for \( x \leq 1 \) to make \( f \) defined on \( (0, \infty) \). Then \( f \) is not slow, as \( f(2x_n) \geq ef(x_n) \) for all \( n \geq 0 \). Pick any \( x > 1 \). If \( x \in (x_n, x_{n+1}] \), then \( x \geq e^{n^2} \) and so \( n \leq \sqrt{\log x} \) and so \( f(x) = e^{n^2} \leq e^{\sqrt{\log x}} \). Although \( f \) is not slow, it
is bounded above by a function which is slow (by (b)), and thus \( \lim_{x \to \infty} \frac{f(x)}{x^a} = 0 \) for every \( a > 0 \) (by (d)). (It is of course also possible to show directly that \( e^{\sqrt{\log x}/x^a} \) goes to 0.)

4. The result is trivially true for \( k = 0 \). For \( k - 1 \to k \) step, assume \( k \geq 1 \) and that there exist \( x_1 < \ldots < x_{k+1} \) so that \( f(x_1) = \ldots = f(x_{k+1}) = 0 \). By the Mean Value Theorem, there exist \( c_i \in (x_i, x_{i+1}) \), so that \( f'(c_i) = 0 \), \( i = 1, \ldots, k \). By the induction hypothesis applied to the function \( g = f' \), there exists \( z \in \mathbb{R} \) such that \( g^{(k-1)}(z) = 0 \), but \( g^{(k-1)} = f^{(k)} \).

5. (a) This series converges uniformly on \( \mathbb{R} \) as

\[
\left| \frac{\sin(nx)}{n^2} \right| \leq \frac{1}{n^2}.
\]

(b) This series converges uniformly on every bounded interval. For example, for \( x \in [-M, M] \),

\[
\left| \frac{\log(1 + nx^2)}{n^2} \right| \leq \frac{\log(1 + nM^2)}{n^2},
\]

and the series with these terms converges, say by comparison to \( n^{-3/2} \).

(c) It is not true that the series converges uniformly on \( \mathbb{R} \). To show this, let \( f_N(x) \) be the sum of first \( N \) terms. We need to show that \( r_N = \sup_{x \in \mathbb{R}} |f(x) - f_N(x)| \) does not converge to 0 as \( N \to \infty \). Now, for every \( n \) one can choose \( x_n \) so that \( \log(1 + nx^2_n) \geq n^3 \). Then

\[
r_N \geq f(x_{N+1}) - f_N(x_{N+1}) = \sum_{n=N+1}^{\infty} \frac{\log(1 + nx^2_{N+1})}{n^2} \geq \frac{\log(1 + (N+1)x^2_{N+1})}{(N+1)^2} \geq N + 1 \to \infty.
\]

6. (a) This is an easy application of ratio test.

(b) This follows from the fact that derivative of a power series within its radius of convergence can be obtained through term-by-term differentiation, and some algebra.

7. (a) That \( d \) is positive and symmetric is clear, and so is the fact that \( d(a, b) = 0 \) if and only if \( a = b \). To prove the triangle inequality, assume \( a, b, c \in S \). We need to prove \( d(a, c) \leq d(a, b) + d(b, c) \). We may assume \( a \neq b \) and \( b \neq c \) or else this is trivial. So assume \( N(a, b) = i \) and \( N(b, c) = j \), and (without loss of generality) that \( i \leq j \). Then \( N(a, c) \geq i \) and so \( d(a, c) \leq 2^{-i} < 2^{-i} + 2^{-j} = d(a, b) + d(b, c) \).

(b) Assume \( a_n = (0.0111\ldots) \), with \( n - 1 \) leading 0s. Then \( d(a_n, z) = 2^{-n} \).