Math 125B, Winter 2015.

Discussion problems 1

Note. These are 25/125A review problems, mostly collected from past exams.

1. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous, and $f(x) > 0$ for all $x \in [0, 1]$. (a) Show that the function $g : [0, 1] \to \mathbb{R}$ given by $g(x) = 1/f(x)$ is bounded. (b) Does the result from (a) remain true if the closed interval $[0, 1]$ is replaced by the open interval $(0, 1)$?

2. Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous functions. (a) Must their product fg be uniformly continuous? (b) Answer the question in (a) if both f and g are uniformly continuous and bounded.

3. Call $f : (0, \infty) \to (0, \infty)$ *slow* if f is nondecreasing and $\lim_{x\to\infty} \frac{f(2x)}{f(x)} = 1$. (a) Give an example of a slow function. (b) Is f given by $f(x) = e^{\sqrt{\log x}}$ slow? (c) Show that if f is slow, then $\lim_{x\to\infty} \frac{f(ax)}{f(x)} = 1$ for every $a > 0$. (d) Show that if f is slow, then $\lim_{x\to\infty} \frac{f(x)}{x^a} = 0$ for every $a > 0$. (e) Show that the reverse implication from (d) does not hold.

4. Assume $k \geq 0$ and that $f : \mathbb{R} \to \mathbb{R}$ is such its kth derivative $f^{(k)}$ exists and is continuous everywhere. Assume that there exists $k + 1$ different numbers $x_1, \ldots, x_{k+1}$ so that $f(x_1) = \cdots = f(x_{k+1}) = 0$. Show that $f^{(k)}(z) = 0$ for some $z \in \mathbb{R}$. Give a careful inductive proof.

5. Show that the functions (a) $f(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ and (b) $f(x) = \sum_{n=1}^{\infty} \frac{\log(1+nx^2)}{n^2}$ are defined and continuous everywhere. Does the series in (b) converge uniformly on $\mathbb{R}$?

6. Let $f(x) = 1 + \sum_{n=1}^{\infty} a_n x^{3n}$, where

$$a_n = \frac{1 \cdot 4 \cdot 7 \cdot 10 \cdots (3n-2)}{(3n)!}.$$

(a) Prove that the series converges for every $x \in \mathbb{R}$. (b) Show that $f''(x) = xf(x)$ for every $x \in \mathbb{R}$.

7. Let S be the set of all binary sequences $\{a = (\alpha_1\alpha_2, \ldots) : \alpha_i \in \{0, 1\} \text{ for all } i\}$. For $a = (\alpha_1\alpha_2, \ldots) \in S$ and $b = (\beta_1\beta_2, \ldots) \in S$, $a \neq b$, let $N(a, b)$ be the smallest $i \in \mathbb{N}$ so that $\alpha_i \neq \beta_i$. Define $d : S \times S \to \mathbb{R}$ by setting

$$d(a, b) = \begin{cases} 2^{-N(a,b)} & \text{if } a \neq b \\ 0 & \text{otherwise} \end{cases}$$

(a) Prove that d is a metric on S. (b) Let $z = (000\ldots)$ be the zero sequence. Give an example of a sequence $a_n \in S$, $a_n \neq z$ for all n, so that $\lim a_n = z$.

Brief solutions

1. (a) Let $m = \inf\{f(x) : x \in [0,1]\}$. As f is continuous on a compact interval, it attains its minimum at an $x_0 \in [0,1]$. Thus $m = f(x_0) > 0$. It follows that $m \le f(x)$ and $0 < 1/f(x) \le 1/m$ for every $x \in [0,1]$.
(b) No. Take $f(x) = x$ as a counterexample.

2. (a) No. Take $f(x) = g(x) = x$. The product is $h(x) = x^2$ which is not uniformly continuous: for $\delta > 0$, $h(n+\delta) - h(n) \ge 2n\delta \ge 2$ if $n \ge 1/\delta$.
(b) Yes. By the assumption, there exists an $M \in \mathbb{R}$ so that $f(x) \le M$ and $g(x) \le M$ for all $x \in \mathbb{R}$. Pick an $\epsilon > 0$. Then pick a $\delta > 0$ so that $|x_1 - x_2| < \delta$ implies $|f(x_2) - f(x_1)| < \epsilon/(2M)$ and $|g(x_2) - g(x_1)| < \epsilon/(2M)$, which can be done as f and g are uniformly continuous. Then, if $|x_1 - x_2| < \delta$,

$$\begin{aligned}|f(x_2)g(x_2) - f(x_1)g(x_1)| &= |f(x_2)g(x_2) - f(x_1)g(x_2) + f(x_1)g(x_2) - f(x_1)g(x_1)| \\ &\le |f(x_2) - f(x_1)||g(x_2)| + |f(x_1)||g(x_2) - g(x_1)| \\ &\le M|f(x_2) - f(x_1)| + M|g(x_2) - g(x_1)| < \epsilon.\end{aligned}$$

3. (a) The simplest example is $f(x) = \log(x+1)$.
(b) Yes, as

$$\sqrt{\log(2x)} = \sqrt{\log x} + \frac{\log 2}{\sqrt{\log(2x)} + \sqrt{\log x}}$$

and the last summand goes to 0.
(c) We may assume that $a \ge 1$: if $a < 1$ we flip the fraction and make the substitution $ax = t$. Let us first assume $a = 2^k$ for some $k > 1$. Then

$$\frac{f(ax)}{f(x)} = \frac{f(2x)}{f(x)}\frac{f(2^2x)}{f(2x)}\cdots\frac{f(2^kx)}{f(2^{k-1}x)} \to 1$$

as every one of the k factors converges to 1. Now for any $a > 1$ there exists a k so that $a \le 2^k$, and then by monotonicity

$$1 \le \frac{f(ax)}{f(x)} \le \frac{f(2^kx)}{f(x)}$$

and the result follows by the sandwich theorem.
(d) Fix an $a > 0$ and pick an $\epsilon > 0$ so that $1 + \epsilon < 2^{a/2}$. As f is slow, then there is an x_0 so that, for $x \ge x_0$, $f(2x) \le f(x)(1+\epsilon)$. Therefore, for each integer $k \ge 0$

$$f(2^kx_0) \le (1+\epsilon)f(2^{k-1}x_0) \le (1+\epsilon)^2 f(2^{k-2}x_0) \le \cdots \le (1+\epsilon)^k f(x_0) \le 2^{ak/2}f(x_0).$$

Take an $x \ge x_0$. Then there exists a an integer $k \ge 1$ so that $2^{k-1}x_0 \le x < 2^kx_0$ and then by monotonicity

$$f(x) \le f(2^kx_0) \le (2^k)^{a/2}f(x_0) \le (2x_0^{-1}x)^{a/2}f(x_0) = 2^{a/2}x_0^{-a/2}f(x_0) \cdot x^{a/2}.$$

Clearly, divided by x^a, the last expression goes to 0 as $x \to \infty$.
(e) Let $x_n = e^{n^2}$ for $n \ge 0$. Then let $f(x) = e^n$ for $x \in (x_n, x_{n+1}]$. Also let $f(x) = 1$ for $x \le 1$ to make f defined on $(0,\infty)$. Then f is not slow, as $f(2x_n) \ge ef(x_n)$ for all $n \ge 0$. Pick any $x > 1$. If $x \in (x_n, x_{n+1}]$, then $x \ge e^{n^2}$ and so $n \le \sqrt{\log x}$ and so $f(x) = e^n \le e^{\sqrt{\log x}}$. Although f is not slow, it

is bounded above by a function which is slow (by (b)), and thus $\lim_{x\to\infty} \frac{f(x)}{x^a} = 0$ for every $a > 0$ (by (d)). (It is of course also possible to show directly that $e^{\sqrt{\log x}}/x^a$ goes to 0.)

4. The result is trivially true for $k = 0$. For $k-1 \to k$ step, assume $k \geq 1$ and that there exist $x_1 < \ldots < x_{k+1}$ so that $f(x_1) = \cdots = f(x_{k+1}) = 0$. By the Mean Value Theorem, there exist $c_i \in (x_i, x_{i+1})$, so that $f'(c_i) = 0$, $i = 1, \ldots, k$. By the induction hypothesis applied to the function $g = f'$, there exists $z \in \mathbb{R}$ such that $g^{(k-1)}(z) = 0$, but $g^{(k-1)} = f^{(k)}$.

5. (a) This series converges uniformly on $\mathbb{R}$ as

$$\left|\frac{\sin(nx)}{n^2}\right| \leq \frac{1}{n^2}.$$

(b) This series converges uniformly on every bounded interval. For example, for $x \in [-M, M]$,

$$\left|\frac{\log(1+nx^2)}{n^2}\right| \leq \frac{\log(1+nM^2)}{n^2},$$

and the series with these terms converges, say by comparison to $n^{-3/2}$.
(c) It is not true that the series converges uniformly on $\mathbb{R}$. To show this, let $f_N(x)$ be the sum of first N terms. We need to show that $r_N = \sup_{x\in\mathbb{R}} |f(x) - f_N(x)|$ does not converge to 0 as $N \to \infty$. Now, for every n one can choose x_n so that $\log(1 + nx_n^2) \geq n^3$. Then

$$r_N \geq f(x_{N+1}) - f_N(x_{N+1}) = \sum_{n=N+1}^{\infty} \frac{\log(1+nx_{N+1}^2)}{n^2} \geq \frac{\log(1+(N+1)x_{N+1}^2)}{(N+1)^2} \geq N+1 \to \infty.$$

6. (a) This is an easy application of ratio test.
(b) This follows from the fact that derivative of a power series within its radius of convergence can be obtained through term-by-term differentiation, and some algebra.

7. (a) That d is positive and symmetric is clear, and so is the fact that $d(a,b) = 0$ if and only if $a = b$. To prove the triangle inequality, assume $a, b, c \in S$. We need to prove $d(a,c) \leq d(a,b) + d(b,c)$. We may assume $a \neq b$ and $b \neq c$ or else this is trivial. So assume $N(a,b) = i$ and $N(b,c) = j$, and (without loss of generality) that $i \leq j$. Then $N(a,c) \geq i$ and so $d(a,c) \leq 2^{-i} < 2^{-i} + 2^{-j} = d(a,b) + d(b,c)$.
(b) Assume $a_n = (0\ldots0111\ldots)$, with $n-1$ leading 0s. Then $d(a_n, z) = 2^{-n}$.