Math 125B, Winter 2015.

Discussion problems 10

Note. These are problems on multidimensional integration.

1. Let $R = [0, 1] \times [0, 1]$ and assume $f : R \to \mathbb{R}$ be a bounded function. For each statement below determine, with proof, whether it is true or false for every such f.
(a) If $g(x) = \int_0^1 f(x, y)\, dy$ exists for every x, $h(y) = \int_0^1 f(x, y)\, dx$ exists for every y, and $\int_0^1 g(x)\, dx$ and $\int_0^1 h(y)\, dy$ both exist, then $\int_R f$ exists.
(b) If $\int_R f$ exists, then $g(x) = \int_0^1 f(x, y)\, dy$ exists for every x.
(c) If f is continuous, $A \subseteq R$, and $\int_{\bar{A}} f$ exists, then so does $\int_A f$.
(d) If $\int_R f$ exists, and $D = \{(x, y) \in R : x = y\}$ then $\int_D f = 0$.

2. Evaluate the iterated integral

$$\int_0^1 dx \int_{\sqrt{x}}^1 \sin\left(\frac{y^3 + 1}{2}\right) dy.$$

3. Let $R = [0, 1] \times [0, 1]$ and $f : R \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{(x^2 + y^2)^2} & x \neq 0 \text{ and } y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that both iterated integrals

$$\int_0^1 dx \int_0^1 f(x, y)\, dy, \quad \int_0^1 dx \int_0^1 f(x, y)\, dy$$

exist (where all integrals are interpreted as one-dimensional Riemann integrals), but they are not equal.

4. Let $R = [-1, 1] \times [-1, 1]$ and $f : R \to \mathbb{R}$ given by

$$f(x, y) = \begin{cases} \frac{xy}{(x^2 + y^2)^2} & (x, y) \neq (0, 0) \\ 0 & \text{otherwise} \end{cases}$$

Show that both iterated integrals

$$\int_{-1}^1 dx \int_{-1}^1 f(x, y)\, dy, \quad \int_{-1}^1 dx \int_{-1}^1 f(x, y)\, dy$$

exist (where all integrals are interpreted as one-dimensional Riemann integrals) and are equal, but $\int_R f$ does not exist. In fact, show that even $\int_{(0,1)\times(0,1)} f$ does not exists in the improper sense.

5. Let $A \subset \mathbb{R}^3$ be given by

$$A = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 \le 1 \text{ and } x_1^2 + x_3^2 \le 1\}.$$

(This set is the intersection of two solid cylinders.) Prove that A is Jordan measurable and compute $\mathrm{Vol}_3(A)$. Carefully verify that hypotheses of relevant theorems apply.

6. Let $f(x, y) = 1/(y + 1)^2$, $A = \{(x, y) \in \mathbb{R}^2 : x > 0, x < y < 2x\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x > 0, x^2 < y < 2x^2\}$. For each of the two improper integrals $\int_A f$ and $\int_B f$, verify if it exists and, if it does, compute it (by computer, if possible).

7. *Bipolar coordinates in* $\mathbb{R}^4$. Consider the coordinate change in $\mathbb{R}^4$ given by $(x_1, x_2, x_3, x_4) = (r_1 \cos\theta_1, r_1 \sin\theta_1, r_2 \cos\theta_2, r_2 \sin\theta_2)$, where (r_1, θ_1) are polar coordinates in x_1x_2-plane and (r_2, θ_2) are polar coordinates in x_3x_4-plane. Let $S = \{(x_1, x_2, x_3, x_4) : x_1^2 + x_2^2 \le 1, x_3^2 + x_4^2 \le 1\}$. Compute the four-dimensional integral

$$\int_S x_1^2 \, dx.$$

8. Let $S = \{(x, y) : x \ge 0, y \ge 0, x + y \le 1\}$. Let

$$f(x, y) = \begin{cases} e^{\frac{y-x}{y+x}} & x + y \ne 0 \\ 0 & \text{otherwise} \end{cases}$$

Show that that f is Riemann integrable on S and compute $\int_S f$.

Brief solutions

1. (a) No, let f be the characteristic function of the set in Problem 1 on Discussion 6. Both iterated integrals are clearly 0, yet the set and its complement are dense in R.

(b) No. For example, $f(x,0)$ could be Dirichlet function and $f(x,y)=0$ when $y>0$. This is an integrable function, with $\int_R f = 0$.

(c) No. For example f is constant 1 on R, and A the set of rational points in R. Then $\bar{A}=R$, but $f\chi_A$ is not integrable on R.

(d) Yes. The diagonal line D (as a graph of a continuous function) has measure 0. It is also a closed set, thus Jordan measurable. Thus $f\chi_D$ is integrable and thus $\int_D f = 0$.

2. Let $S=\{(x,y)\in\mathbb{R}^2 : x\in[0,1], \sqrt{x}\le y\le 1\}$. This is clearly a Jordan region and $f(x,y)=\sin\left(\frac{y^3+1}{2}\right)$ is a continuous function on S. Therefore the integral above is $\int_S f$ and we may evaluate it in any order. Reversing the order, we get,

$$\int_0^1 dy \int_0^{y^2} \sin\left(\frac{y^3+1}{2}\right) dx = \int_0^1 \sin\left(\frac{y^3+1}{2}\right) y^2\,dy = \frac{2}{3}\int_{1/2}^1 \sin z\,dz = \frac{2}{3}\left(\cos(1/2)-\cos 1\right).$$

3. Consider

$$\int_0^1 dx \int_0^1 f(x,y)\,dy.$$

We will show that the integral $g(x)=\int_0^1 f(x,y)\,dy$ exists for all $x\in[0,1]$. Note that the existance is clear when $x=0$ and $g(0)=0$.

Now assume $x>0$. Then the function $y\mapsto f(x,y)$ is not continuous at $y=0$, but becomes continuous and differentiable everywhere if we redefine $f(x,0)=1/x^2$, which we may do for the purpose of existence of computation of Riemann integral $\int_0^1 f(x,y)\,dy$. Then $f(x,y)=\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right)$, and so $g(x)=\frac{1}{x^2+1}$. Thus

$$g(x)=\begin{cases}\frac{1}{x^2+1} & x\neq 0\\ 0 & x=0\end{cases}$$

and is clearly a bounded function on $[0,1]$, with

$$\int_0^1 g(x)\,dx = \frac{\pi}{4}.$$

The other iterated integral equals $-\pi/4$, by the fact that $f(y,x)=-f(x,y)$.

4. By symmetry, the two iterated integrals are equal, if they exist. Consider

$$\int_{-1}^1 dx \int_{-1}^1 f(x,y)\,dy.$$

Note that $f(x,y)=0$ whenever $y=0$. Thus, for all $x\in[-1,1]$, the integral $g(x)=\int_0^1 f(x,y)\,dy$ exists as a Riemann integral of a continuous function, and is in fact equal to 0 as integral of an odd function. Thus $g(x)=0$ for every x and both iterated integrals are 0.

Clearly f is not bounded, so $\int_R f$ cannot exist as a proper Riemann integral. For $\int_{(0,1)\times(0,1)} f$, pick a small $\epsilon > 0$ and consider $\int_{(\epsilon,1)\times(\epsilon,1)} f = \int_{[\epsilon,1]\times[\epsilon,1]} f$, which can be evaluated by Fubini, with dy integration first, to get

$$\begin{aligned}\int_{[\epsilon,1]\times[\epsilon,1]} f &= \frac{1}{2}\int_{[\epsilon,1]} \left(\frac{x}{x^2+\epsilon^2} - \frac{x}{x^2+1}\right) dx \\ &= -\frac{1}{4}\left(\log(2\epsilon^2) + \log 2 - 2\log(1+\epsilon^2)\right),\end{aligned}$$

which clearly goes to ∞ as $\epsilon \to 0$.

5. Let $b(x) = \sqrt{1-x^2}$. Then

$$A = \{(x_1,x_2,x_3) : x_1 \in [0,1], x_2 \in [-b(x_1), b(x_1)], x_3 \in [-b(x_1), b(x_1)]\}.$$

Now $[0,1] \subset \mathbb{R}$ is Jordan measurable, so

$$A_1 = \{(x_1,x_2) : x_1 \in [0,1], x_2 \in [-b(x_1), b(x_1)]\} \subset \mathbb{R}^2$$

is Jordan measurable, as it is the set between two graphs of continuous functions ($-b$ and b) on a Jordan measurable set $[0,1]$. Then, with $b_1 : \mathbb{R}^2 \to \mathbb{R}$ given by $b_1(x_1,x_2) = b(x_1)$,

$$A = \{(x_1,x_2,x_3) : (x_1,x_2) \in A_1, x_3 \in [-b_1(x_1,x_2), b_1(x_1,x_2)]\}$$

is Jordan measurable , as it is the set between two graphs of continuous functions ($-b_1$ and b_1) on a Jordan measurable set A_1. By the iterated integrals theorem,

$$\begin{aligned}\mathrm{Vol}_3(A) = \int_A \chi_A &= \int_{-1}^1 dx_1 \int_{-1}^1 dx_2 \int_{-1}^1 \chi_A(x_1,x_2,x_3)\, dx_3 \\ &= \int_{-1}^1 dx_1 \int_{-b(x_1)}^{b(x_1)} dx_2 \int_{-b(x_1)}^{b(x_1)} dx_3 \\ &= 2\int_{-1}^1 dx_1 \int_{-b(x_1)}^{b(x_1)} b(x_1)\, dx_2 \\ &= 4\int_{-1}^1 b(x_1)^2\, dx_1 = \frac{16}{3}.\end{aligned}$$

6. Let $A_N = \{(x,y) \in \mathbb{R}^2 : x \in (0,N), x < y < 2x\}$. Then

$$\int_{A_N} f = \int_0^N dx \int_x^{2x} f(x,y)\, dy = \int_0^N \frac{x}{(x+1)(2x+1)}\, dx,$$

which goes to ∞ as $N \to \infty$ by comparison with $1/x$. Analogously, we define B_N and get

$$\int_{B_N} f = \int_0^N \frac{x^2}{(x^2+1)(2x^2+1)}\, dx,$$

which converges as $N \to \infty$ by comparison with $1/x^2$. The exact limit is $(1-\sqrt{2}/2)\pi/4$.

7. The Jacobian is $r_1 r_2$ and so the integral equals

$$\int_0^{2\pi} \cos^2\theta_1\, d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^1 r_1^2 \cdot r_1\, dr_1 \int_0^1 r_2\, dr_2 = \frac{\pi^2}{4}.$$

8. Note that f is discontinuous at a single point $(0,0)$ of S. Also, $|f| \le 1$ on S: if $x \le y$, then $|f| = (y-x)/(y+x) \le y/y = 1$. Riemann integrability follows.

Moreover, by change of variables $u = y - x$, $v = y + x$, we get $(x, y) = ((v-u)/2, (v+u)/2)$, which is linear change with matrix

$$T = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

and $\det T = -1/2$. Further, $T^{-1}S = \{(u,v) : v \in [0,1], -v \le u \le v\}$. By the change of variables theorem, our integral equals

$$\frac{1}{2}\int_{T^{-1}S} e^{u/v}\,du\,dv = \frac{1}{2}\int_0^1 dv \int_{-v}^{v} e^{u/v}\,du = \frac{1}{2}\int_0^1 (e-1/e)v\,dv = \frac{1}{4}(e-1/e).$$