Math 125B, Winter 2015.

Discussion problems 2

Note. These are problems on the definition of Riemann integral and integrable functions.

1. Assume that $f : [0,1] \to \mathbb{R}$ is bounded. Determine, with proof, whether each statement below is true or false.
(a) If $U(f,P) = L(f,P)$ for some partition P of $[0,1]$, then f is constant.
(b) If f is continuous on $(0,1)$, then it is integrable.
(c) If f is continuous on $[0,0.99]$, then it is integrable.
(d) If $\frac{1}{n}\sum_{k=1}^{n} f(k/n)$ converges as $n \to \infty$, then f is integrable.
(e) If f is integrable, then $\frac{1}{n}\sum_{k=1}^{n} f(k/n)$ converges as $n \to \infty$.
(f) If f is integrable, $U(f,P_n)$ converges to $\int_a^b f(x)\,dx$ for any sequence of partitions P_n whose norms go to 0.

2. Suppose that $f : [a,b] \to \mathbb{R}$, $f \geq 0$ but not necessarily bounded.
(a) Show that $L(f,P)$ is finite for any partition P.
(b) Show that for $f : [0,1] \to \mathbb{R}$, given by $f(x) = \begin{cases} 1/x & x > 0 \\ 0 & x = 0 \end{cases}$, the supremum over all partitions $\sup_P L(f,P)$ equals ∞.

Brief solutions

1. (a) Yes. Assume $0 = x_0 < x_1 < \cdots < x_n = 1$ is a partition and $L(f,P) = U(f,P)$. Then $\inf_{[x_{j-1},x_j]} f = \sup_{[x_{j-1},x_j]} f$ for $j = 1,\dots,n$, and so f is a constant c_j on $[x_{j-1},x_j]$. For $j = 1,\dots,n-1$, $[x_{j-1},x_j]$ and $[x_j,x_{j+1}]$ share a point and so $c_j = c_{j+1}$. Thus $c_1 = c_2 = \cdots = c_n$ and f is constant.

(b) Yes. Assume $|f| \le M$. Pick an $\epsilon > 0$. Then f is integrable on $[\epsilon/(8M), 1-\epsilon/(8M)]$ (as it is continuous there) and so there exists a partition P' of $[\epsilon/(8M), 1-\epsilon/(8M)]$ so that $U(f,P')-L(f,P') < \epsilon/2$. Form the partition $P = P' \cup \{0,1\}$ of $[0,1]$. Let P be given by n points

$$0 = x_0 < \epsilon/(8M) = x_1 < x_2 < \cdots < 1-\epsilon/(8M) = x_{n-1} < x_n = 1.$$

Then on the first interval given by the partition P $(\sup_{[x_0,x_1]} f - \inf_{[x_0,x_1]} f)\Delta x_1 < 2M \cdot \epsilon/(8M) = \epsilon/4$ and the analogous inequality holds for the last interval. Thus

$$U(f,P) - L(f,P) \le 2\epsilon/4 + U(f,P') - L(f,P') < \epsilon/2 + \epsilon/ = \epsilon.$$

(c) No. Fix any $\beta < 1$, in our case we can take, say, $\beta = 0.991$. Take a function f such that is $f(x) = 0$ unless x is a rational number in $[\beta,1]$, in which case $f(x) = 1$. This is a continuous function on $[0,\beta)$. For any partition P, $\sup f = 1$ and $\inf f = 0$, on all intervals that intersect $[\beta,1]$. Thus $L(f,P) = 0$ and $U(f,P) \ge 1-\beta$.

(d) No. Take Dirichlet function f, for which $f(x) = 1$ when x is rational and 0 otherwise. The sum then equals 1 for every n (as $k/n \in \mathbb{Q}$), but f is not integrable.

(e) Yes. The sum is a Riemann sum with evaluations at right endpoints and norm $1/n \to 0$. Thus it converges to $\int_0^1 f$.

(f) Yes. For an partition P_n with m intervals, and for any interval I_j generated by P_n, choose $c_j \in I_j$ so that $f(c_j) \ge \sup_{I_j} f - 1/n$. Thus

$$\sum_{j=1}^{m} f(c_j)\,\Delta x_j \ge \sum_{j=1}^{m} \sup_{I_j} f\,\Delta x_j - \sum_{j=1}^{m} \frac{1}{n}\,\Delta x_j = U(f,P_n) - \frac{1}{n}.$$

Therefore

$$\sum_{j=1}^{m} f(c_j)\,\Delta x_j \le U(f,P_n) \le \sum_{j=1}^{m} f(c_j)\,\Delta x_j + \frac{1}{n}$$

and the lower bound and the upper bound both converge to $\int_0^1 f$.

2. (a) Observe that $\inf_{I_j} f$ is finite for any I_j as f is bounded below by 0. Thus $L(f,P)$ is finite.

(b) Take equidistant partition of $[0,1]$ into n intervals. For $j \ge 2$, the minimum on $[(j-1)/n, j/n]$ is n/j, while for $j = 1$ it is 0. Thus $L(f,P) = \sum_{j=2}^{n} \frac{n}{j} \cdot \frac{1}{n} = \sum_{j=2}^{n} \frac{1}{j} \to \infty$, by divergence of the harmonic series.