Discussion problems 2

Note. These are problems on the definition of Riemann integral and integrable functions.

1. Assume that $f : [0, 1] \rightarrow \mathbb{R}$ is bounded. Determine, with proof, whether each statement below is true or false.
   (a) If $U(f, P) = L(f, P)$ for some partition $P$ of $[0, 1]$, then $f$ is constant.
   (b) If $f$ is continuous on $(0, 1)$, then it is integrable.
   (c) If $f$ is continuous on $[0, 0.99]$, then it is integrable.
   (d) If $\frac{1}{n} \sum_{k=1}^{n} f(k/n)$ converges as $n \to \infty$, then $f$ is integrable.
   (e) If $f$ is integrable, then $\frac{1}{n} \sum_{k=1}^{n} f(k/n)$ converges as $n \to \infty$.
   (f) If $f$ is integrable, $U(f, P_n)$ converges to $\int_{a}^{b} f(x) \, dx$ for any sequence of partitions $P_n$ whose norms go to 0.

2. Suppose that $f : [a, b] \rightarrow \mathbb{R}$, $f \geq 0$ but not necessarily bounded.
   (a) Show that $L(f, P)$ is finite for any partition $P$.
   (b) Show that for $f : [0, 1] \rightarrow \mathbb{R}$, given by $f(x) = \begin{cases} 1/x & x > 0 \\ 0 & x = 0 \end{cases}$, the supremum over all partitions $\sup_{P} L(f, P)$ equals $\infty$. 
Brief solutions

1. (a) Yes. Assume \( 0 = x_0 < x_1 < \cdots < x_n = 1 \) is a partition and \( L(f, P) = U(f, P). \) Then \( \inf_{[x_{j-1}, x_j]} f = \sup_{[x_{j-1}, x_j]} f \) for \( j = 1, \ldots, n, \) and so \( f \) is a constant \( c_j \) on \( [x_{j-1}, x_j]. \) For \( j = 1, \ldots, n-1, \) \( [x_{j-1}, x_j] \) and \( [x_j, x_{j+1}] \) share a point and so \( c_j = c_{j+1}. \) Thus \( c_1 = c_2 = \cdots = c_n \) and \( f \) is constant.

(b) Yes. Assume \( |f| \leq M. \) Pick an \( \epsilon > 0. \) Then \( f \) is integrable on \( [\epsilon/(8M), 1 - \epsilon/(8M)] \) (as it is continuous there) and so there exists a partition \( P' \) of \( [\epsilon/(8M), 1 - \epsilon/(8M)] \) so that \( U(f, P') - L(f, P') < \epsilon/2. \) Form the partition \( P = P' \cup \{0, 1\} \) of \([0, 1].\) Let \( P \) be given by \( n \) points

\[
0 = x_0 < \epsilon/(8M) = x_1 < x_2 < \cdots < 1 - \epsilon/(8M) = x_{n-1} < x_n = 1.
\]

Then on the first interval given by the partition \( P \) \( \sup_{[x_0, x_1]} f - \inf_{[x_0, x_1]} f \Delta x_1 < 2M \cdot \epsilon/(8M) = \epsilon/4 \) and the analogous inequality holds for the last interval. Thus

\[
U(f, P) - L(f, P) \leq 2\epsilon/4 + U(f, P') - L(f, P') < \epsilon/2 + \epsilon/\epsilon = \epsilon.
\]

(c) No. Fix any \( \beta < 1, \) in our case we can take, say, \( \beta = 0.991. \) Take a function \( f \) such that \( f(x) = 0 \) unless \( x \) is a rational number in \( [\beta, 1], \) in which case \( f(x) = 1. \) This is a continuous function on \([0, \beta].\)

For any partition \( P, \) \( \sup f = 1 \) and \( \inf f = 0, \) on all intervals that intersect \( [\beta, 1]. \) Thus \( L(f, P) = 0 \) and \( U(f, P) \geq 1 - \beta. \)

(d) No. Take Dirichlet function \( f, \) for which \( f(x) = 1 \) when \( x \) is rational and 0 otherwise. The sum then equals 1 for every \( n \) (as \( k/n \in \mathbb{Q}, \) but \( f \) is not integrable.

(e) Yes. The sum is a Riemann sum with evaluations at right endpoints and norm \( 1/n \to 0. \) Thus it converges to \( \int_0^1 f. \)

(f) Yes. For an partition \( P_n \) with \( m \) intervals, and for any interval \( I_j \) generated by \( P_n, \) choose \( c_j \in I_j \) so that \( f(c_j) \geq \sup_{I_j} f - 1/n. \) Thus

\[
\sum_{j=1}^{m} f(c_j) \Delta x_j \geq \sum_{j=1}^{m} \sup_{I_j} f \Delta x_j - \sum_{j=1}^{m} \frac{1}{n} \Delta x_j = U(f, P_n) - \frac{1}{n}.
\]

Therefore

\[
\sum_{j=1}^{m} f(c_j) \Delta x_j \geq U(f, P_n) \geq \sum_{j=1}^{m} f(c_j) \Delta x_j + \frac{1}{n}
\]

and the lower bound and the upper bound both converge to \( \int_0^1 f. \)

2. (a) Observe that \( \inf_{I_j} f \) is finite for any \( I_j \) as \( f \) is bounded below by 0. Thus \( L(f, P) \) is finite.

(b) Take equidistant partition of \([0, 1] \) into \( n \) intervals. For \( j \geq 2, \) the minimum on \([ (j-1)/n, j/n ] \) is \( n/j, \) while for \( j = 1 \) it is 0. Thus \( L(f, P) = \sum_{j=2}^{n} \frac{n}{j} \cdot \frac{1}{n} = \sum_{j=2}^{n} \frac{1}{j} \to \infty, \) by divergence of the harmonic series.