Math 125B, Winter 2015.

Discussion problems 3

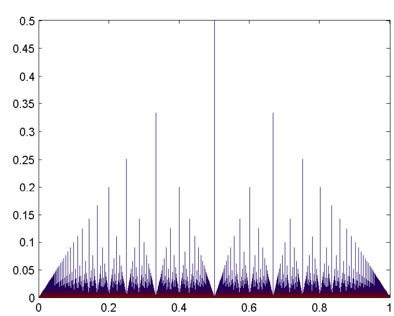
Note. These are problems on the properties of Riemann integral and integrable functions.

1. Assume $f : [a, b] \to \mathbb{R}$ is bounded and integrable, and $[c, d] \subset [a, b]$. Determine, with proof, whether each statement below is true or false.
(a) Is f integrable on $[c, d]$?
(b) If $\int_a^b f \geq \int_c^d f$?
(c) If $f \geq 0$, is $\int_a^b f \geq \int_c^d f$?

2. The *Thomae function* $T : [0, 1] \to \mathbb{R}$ is defined as follows

$$T(x) = \begin{cases} 0 & x \notin \mathbb{Q} \text{ or } x = 0 \text{ or } x = 1 \\ 1/q & x = p/q < 1, \text{ for } p, q \in \mathbb{N} \text{ with } \gcd(p, q) = 1 \end{cases}$$

The figure below is a graph of T. (It is what you would see if you looked from $(0, 0, 0)$ into the "orchard" of "trees" — stalks of equal height — "planted" at all points of $\mathbb{N} \times \mathbb{N} \times \{0\}$.)



(a) Show that T is discontinuous at every rational number in $(0, 1)$ and continuous otherwise.
(b) Show that T is integrable with $\int_0^1 T = 0$.
(c) Define $f : [0, 1] \to \mathbb{R}$ as follows

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$$

Show that f and T are both integrable, but the composite function $f \circ T$ is not.

3. Assume that $f : [a, b] \to \mathbb{R}$ is integrable.
(a) Assume that f is continuous, that $f \geq 0$, (i.e., that $f(x) \geq 0$ for all x) and that $\int_a^b f = 0$. Show that $f = 0$.

(b) Is the conclusion still correct if the continuity assumption is dropped?

(c) Assume that f is continuous, and that $\int_a^b f = \int_a^b |f|$. Show that $f \geq 0$.

4. Assume $f_1, f_2 : [a,b] \to \mathbb{R}$ are integrable and $S \subset [a,b]$ is dense in $[a,b]$. Assume $f_1(x) = f_2(x)$ for every $x \in S$. Prove that $\int_a^b f_1 = \int_a^b f_2$.

5. (a) Assume you have $\ell + 1$ real numbers $a_0 < a_1 < a_2 < \cdots < a_\ell$. Assume that a function $f : [a_0, a_\ell] \to \mathbb{R}$ is bounded and is integrable on any interval $[c,d] \subset [a_0, a_\ell]$ that contains no a_i, $i = 0, \ldots, \ell$. Show that f is integrable on $[a_0, a_\ell]$.

(b) Assume that $f : [a,b] \to \mathbb{R}$ is bounded and its sets of discontinuities is finite. Prove that f is integrable.

6. (a) Assume that $f : [a,b] \to \mathbb{R}$ is integrable, and that $g : [a,b] \to \mathbb{R}$ differs from f at only finitely many points. Show that g is integrable and $\int_a^b f = \int_a^b g$.

(b) Assume that $f : [a,b] \to \mathbb{R}$ is piecewise constant, that is, there exist numbers $a = a_0 < a_1 < a_2 < \cdots < a_\ell = b$ and $c_i \in \mathbb{R}$ so that $f = c_i$ on (a_{i-1}, a_i) for $i = 1, \ldots, \ell$. Show that $\int_a^b f = \sum_{i=1}^{\ell} c_i(a_i - a_{i-1})$.

7. Assume you have n intervals $I_1, \ldots, I_n \subset [0,4]$. They could be open, half-closed, or closed, but the sum of their lengths is at least 125. Show that there exists a point in $[0,1]$ that is covered by at least 32 of these intervals.

Brief solutions

1. (a) Yes. Divide $[a, b] = [a, c] \cup [c, d] \cup [d, b]$. The function is integrable on each of the three intervals (or, more precisely, on those of the three that are nondegenerate) by the domain additivity theorem, proved in the lecture.

(b) No. Take $[a, b] = [-1, 1]$, $[c, d] = [0, 1]$. $f(x) = x$. Then $\int_a^b f = 0$, but $\int_c^d f = 1$

(c) Yes. Divide $[a, b]$ as in (a). Assume all three integrals are nondegenerate, otherwise the proof is even simpler. Then, by domain additivity, $\int_a^b f = \int_a^c f + \int_c^d f + \int_d^b f$ and then by monotonicity $\int_a^c f \geq 0$ and $\int_c^d f \geq 0$, which finishes the proof.

2. (a) Assume that $x \in (0, 1)$ is a rational number. Then $T(x) > 0$ but there is a sequence of irrational numbers x_n with $x_n \to x$ and $T(x_n) = 0$. Thus T is discontinuous at x.

Now assume that x is irrational or 0 or 1. Assume that $x_n \in [0, 1]$ converge to x. We need to show that $T(x_n) \to 0$. At irrational x_n, $T(x_n) = 0$, so we may assume that all x_n are rational. Write $x_n = p_n/q_n$ as a reduced fraction, that is, so that $\gcd(p_n, q_n) = 1$. Then $T(x_n) = 1/q_n$ and we need to show that $q_n \to \infty$. Assume not. Then a subsequence of (x_n), which we may assume to be the full sequence, has bounded q_n, say $2 \leq q_n \leq K$. But there are only finitely many reduced fractions in $[0, 1]$ with denominators at most K; as x_n converges and its elements are chosen from a finite set, it must be eventually constant. But then $x = \lim x_n \in \mathbb{Q}$ and $x \neq 0$, $x \neq 1$. This contradiction shows that $T(x_n) \to 0 = T(x)$ and that T is continuous at x.

(b) As irrational numbers are dense in $[0, 1]$, $L(T, P) = 0$ for any partition P, thus $L(T) = 0$. Pick an $\epsilon > 0$. We will find a partition P so that $U(f, P) < \epsilon$. Note first that $\max f = 1/2$. Pick K large enough so that $1/K < \epsilon/2$. Let S be the set of reduced fractions with denominators at most K, and N the number of elements of S. Choose any partition P, given by $0 = x_0 < x_1 < \cdots < x_n = 1$, with norm $||P|| < \epsilon/(4N)$. Any partition has at most $2N$ intervals which include at least one element of S. Together, those contribute at most $(\max f) \cdot 2N \cdot ||P|| = \epsilon/4$ to $U(f, P)$. The remaining intervals contain no elements of S and so T is bounded above by $1/K$ on each of them; therefore they contribute at most $1/K \sum_{j=1}^n \Delta x_j = 1/K < \epsilon/2$ to $U(f, P)$. Therefore $U(T, P) < 3\epsilon/4 < \epsilon$. It follows that $U(T) \leq U(T, P) < \epsilon$, and then $U(T) = 0 = L(T)$.

(c) We showed in class that $\int_0^1 (1 - f) = 0$, so $f = 1 - (1 - f)$ is integrable with $\int_0^1 f = 1$. (This also follows from Problem 6(b).) We showed in (b) that T is integrable. Finally, $f \circ T(x)$ is 1 when x is a nonzero rational number and 1 otherwise. So it differs from the Dirichlet function only at 0 and 1, and is not integrable by the same argument.

3. (a) If it is not true that $f = 0$, then there exists a number $x_0 \in [a, b]$ so that $f(x_0) > 0$. Let $\alpha = f(x_0)/2$. By continuity, there exists a $\delta > 0$ so that $f(x) \geq \alpha$ for $x \in [x_0 - \delta, x_0 + \delta]$. Then, by Problem 1(c) and monotonicity,

$$\int_a^b f \geq \int_{x_0-\delta}^{x_0+\delta} f \geq \int_{x_0-\delta}^{x_0+\delta} \alpha = 2\delta\alpha > 0.$$

(b) No. The function $f : [0, 1] \to \mathbb{R}$ from Problem 2(c) has $\int_0^1 f = 0$.

(c) Take $g = |f| - f$. Then $g \geq 0$ and $\int_a^b g = 0$. So $g = 0$, and so $f = |f| \geq 0$.

4. Let $R_i = \int_a^b f_i$. Fix an $\epsilon > 0$. Then there is a $\delta > 0$ so that $||P|| < \delta$ implies $|S(f_i, P, C) - R_i| < \epsilon/2$ for every tag set C. For any partition $||P||$, choose C so that all $c_i \in S$, which can be done since S is

dense. Then $S(f_1, P, C) = S(f_2, P, C)$. Then

$$\begin{aligned} |R_1 - R_2| &= |R_1 - S(f_1, P, C) + S(f_2, P, C) - R_2| \\ &\leq |R_1 - S(f_1, P, C)| + |S(f_2, P, C) - R_2| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

As this is true for every $\epsilon > 0$, $R_1 = R_2$.

5. (a) It is enough to show this for $\ell = 2$; the general case then follows by induction. Then this becomes a very similar to Problem 1(b) on Discussion Problems 2, but nevertheless we repeat the proof. Assume $|f| \leq M$. Pick an $\epsilon > 0$. Then f is integrable on $[a_1 + \epsilon/(8M), a_2 - \epsilon/(8M)]$, and so there exists a partition P' of $[\epsilon/(8M), 1 - \epsilon/(8M)]$ so that $U(f, P') - L(f, P') < \epsilon/2$. Form the partition $P = P' \cup \{a_1, a_2\}$ of $[a_1, a_2]$. If $[x_0, x_1]$ is the first interval given by the partition P, we have

$$(\sup_{[x_0,x_1]} f - \inf_{[x_0,x_1]} f)\Delta x_1 < 2M \cdot \epsilon/(8M) = \epsilon/4$$

and the analogous inequality holds for the last interval. Thus $U(f, P) - L(f, P) \leq 2\epsilon/4 + \epsilon/2 = \epsilon$.

(b) Let $a_1, \ldots, a_{\ell-1}$ be the discontinuities of f inside (a, b) and $a_0 = a$, $a_\ell = b$. Then clearly (a) applies.

6. (a) Integrability follows from Problem 5. Equality of integrals follows from Problem 4.

(b) By domain additivity.

$$\int_a^b f = \sum_{i=1}^{\ell} \int_{x_{i-1}}^{x_i} f.$$

Then,

$$\int_{x_{i-1}}^{x_i} f = \int_{x_{i-1}}^{x_i} c_i,$$

by (a), as f and c_i may only differ at the endpoints of $[x_{i-1}, x_i]$. Finally, the last integral of a constant function equals $c_i(x_i - x_{i-1})$.

7. Let $f_k : [0, 4] \to \mathbb{R}$ be given by

$$f_k(x) = \begin{cases} 1 & x \in I_k \\ 0 & \text{otherwise} \end{cases}.$$

These are integrable functions by Problem 5(b). If the claim is not true, $\sum f_k < 32$, so in fact $\sum f_k \leq 31$. Then, by monotonicity, $\int_0^4 \sum f_k \leq 4 \cdot 31 = 124$. By additivity, and because $\int_0^4 f_k = |I_k|$ by Problem 6(b), $\int_0^4 \sum f_k = \sum \int_0^4 f_k = \sum |I_k| \geq 125$, contradiction.