Math 125B, Winter 2015.

Discussion problems 4

Note. These are problems on the theorems about the Riemann integral.

1. Assume $f : [a, b] \to \mathbb{R}$ is Riemann integrable, and $F(x) = \int_a^x f(t)\,dt$. Determine, with proof, whether each statement below is true or false.

(a) If $I_k \subset [a, b]$ is a sequence of closed intervals with $\lim_{k\to\infty} |I_k| \to 0$, then $\lim_{k\to\infty} \int_{I_k} f \to 0$.

(b) F is differentiable for every $x \in (a, b)$

(c) If f is strictly increasing (i.e., $x < y \implies f(x) < f(y)$ for all $x \in [a, b]$), then there exists a unique $x_0 \in [a, b]$ at which F achieves its minimum.

(d) If f is strictly increasing, then there exists a unique $x_0 \in [a, b]$ at which F achieves its maximum.

2. Assume $f : [a, b] \to \mathbb{R}$ is bounded. Assume also that f is continuous on (a, b). Show that there exists a $c \in (a, b)$ so that $f(c) = \frac{1}{b-a} \int_a^b f$.

3. (a) Assume $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$ with $|f'(x)| \le M$ for all $x \in (0, 1)$. Prove that

$$\left| \int_0^1 f(x)\,dx - \frac{1}{n} \sum_{j=1}^{n} f\left(\frac{j}{n}\right) \right| \le \frac{M}{n}.$$

(b) I got this problem from the book by Thomson, Bruckner and Bruckner, but their formulation does not include the assumption that f is continuous on $[0, 1]$. Is the conclusion is still true?

4. Compute

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2}.$$

Make sure to carefully justify all steps!

5. Prove that $0 \le \int_0^1 x\sqrt{\sin x + 3x}\,dx \le 4/5$.

6. Assume that $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$, and that f' exists on $(0, 1)$ and is integrable on $[0, 1]$. Define

$$I(f) = \int_0^1 x^2 f'(x)\,dx + \int_0^1 2x f(x)\,dx.$$

(a) Explain why both intergrals exist.

(b) Assume $f(x) = (1 + x^{2015})^6$. Compute $I(f)$ (no computer, 30 seconds).

(c) Characterize functions f for which $I(f) = 0$.

Brief solutions

1.(b) Yes. Let $M = \sup_{[a,b]} |f|$. Then $|\int_{I_k} f| \le \int_{I_k} |f| \le M|I_k| \to 0$.

(b) No. Let $f(x)$ be 1 when $x > 0$ and -1 otherwise (the sign function). Then $F(x) = |x| - 1$, which is not differentiable at 0.

(c) Yes. Let $x_0 = \sup\{x \in [a,b] : f(x) < 0\}$, with the proviso that $x_0 = a$ if $f \ge 0$ on $[a,b]$. Assume first that $x_0 > a$ and fix a $c \in (a, x_0)$. Then $f(c) < 0$. (Why? If $f(c) \ge 0$, then $f(x) > f(c) \ge 0$ for all $x > c$, so x_0 is not the supremum.) Then, for $a \le x_1 < x_2 \le c$,

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} f \le f(c)(x_2 - x_1) < 0.$$

This proves F is strictly decreasing on $[a,c]$ and, as $c < x_0$ was arbitrary, on $[a, x_0)$. Further, assume $x_0 < b$ and fix a $d \in (x_0, b)$. Then $f(d) > 0$. (Why? If $f(d) \le 0$, then $f(x) < f(d) \le 0$ for all $x < d$, so x_0 is not an upper bound and thus not the supremum.) Then, for $d \le x_1 < x_2 \le b$,

$$F(x_2) - F(x_1) = \int_{x_1}^{x_2} f \ge f(d)(x_2 - x_1) > 0.$$

This proves F is strictly increasing on $[d,b]$ and, as $d > x_0$ was arbitrary, on $(x_0, b]$. We have proved that F is strictly decreasing on on $[a, x_0)$ and strictly increasing on on $(x_0, b]$, so it achieves its minimum at x_0.

(d) No. Take $f(x) = x$ on $[-1, 1]$. Then $F(x) = x^2 - 1$ and F achieves its maximum at -1 and 1.

2. By the Second Fundamental Theorem of Calculus, $F(x) = \int_a^x f$ is continuous on $[a,b]$, and differentiable on (a,b) with $F' = f$. By the Mean Value Theorem (for the *derivative*), applied to F, there is a $c \in (a,b)$, such that

$$F'(c) = \frac{F(b) - f(a)}{b - a},$$

but $F'(c) = f(c)$.

3. (a) By the Mean Value Theorem, for any $x \in [(j-1)/n, j/n]$, there exists a $c \in ((j-1)/n, j/n)$ so that $f(x) - f(j/n) = f'(c)(x - j/n)$. Then $|f(x) - f(j/n)| = |f'(c)|\,|x - j/n| \le M|x - j/n| \le M/n$. Therefore

$$\left| \int_{[x_{j-1}, x_j]} (f(x) - f(j/n))\, dx \right| \le \int_{[x_{j-1}, x_j]} |f(x) - f(j/n)|\; dx \le \frac{M}{n^2}$$

and so the expression in question can be bounded by

$$\left| \sum_{j=1}^{n} \int_{[x_{j-1}, x_j]} (f(x) - f(j/n))\, dx \right| \le \sum_{j=1}^{n} \left| \int_{[x_{j-1}, x_j]} (f(x) - f(j/n))\, dx \right| \le n \cdot \frac{M}{n^2}.$$

(b) No. While f is bounded (for $x \in (0,1)$, $|f(x) - f(1/2)| \le M|x - 1/2| \le M/2$ and so $|f(x)| \le 3M/2$), and so $\int_0^1 f(x)\, dx$ exists (Problem 1(b) in Discussion Problems 2), we have no control over $f(1)$ which can be changed to become arbitrarily large without affecting either $\int_0^1 f(x)\, dx$ (by Problem 6(a) in Discussion Problems 3) or f' on $(0,1)$, but the sum becomes arbitrarily large!

4. The sum

$$\sum_{i=1}^{n} \frac{n}{n^2 + i^2} = \sum_{i=1}^{n} \frac{1}{1 + (i/n)^2} \cdot \frac{1}{n}$$

is a Riemann sum for $\int_0^1 1/(1+x^2)\,dx$, with equidistant partition into n intervals and tags at right endpoints. As $f(x) = 1/(1+x^2)$ is continuous on $\mathbb{R}$, with an antiderivative $\arctan x$, the Fundamental Theorem of Calculus implies that $\int_0^1 f(x)\,dx = \arctan 1 - \arctan 0 = \pi/4$. Therefore the Riemann sums converge to $\pi/4$.

5. Let f be the function in question. Then $f \geq 0$ on $[0,1]$ and so $\int_0^1 f(x)\,dx \geq 0$. Furthermore, $\sin x \leq x$ for all $x \geq 0$ (e.g., by comparing derivatives) and so $f(x) \leq x\sqrt{x+3x} = 2x^{3/2}$. Thus $\int_0^1 f(x)\,dx \leq 2\int_0^1 x^{3/2}\,dx = 4/5$.

6. (a) Products of integrable functions are integrable.

(b) By parts: $I(f) = f(1) = 64$.

(c) By parts: those functions for which $f(1) = 0$.