Math 125B, Winter 2015.

Discussion problems 5

Note. These are final problems on the Riemann integral, focusing particularly on the improper integral.

1. Assume $f : [0, \infty] \to [0, \infty)$ is continuous. For each statement below determine, with proof, whether it is true or false.
(a) If $\sum_{n=1}^{\infty} f(n) < \infty$, then $\int_0^\infty f(x)\,dx$ converges.
(b) If $\int_0^\infty f(x)\,dx$ converges, then f is bounded on $[0, \infty)$.
(c) If $\int_0^\infty f(x)\,dx$ converges and f is nonincreasing (i.e., $0 \le x_1 \le x_2$ implies $f(x_1) \ge f(x_2)$), then $\lim_{x\to\infty} f(x) = 0$.
(d) If $\int_0^\infty f(x)\,dx$ converges, f is differentiable on $(0, \infty)$ and f' is bounded on $(0, \infty)$, then $\lim_{x\to\infty} f(x) = 0$.

2. Prove that $\int_0^\infty \frac{e^{\sin x}\sin x}{x^2+1}\,dx$ converges.

3. For each integral below, determine whether it converges or not.
(a) $\int_0^1 ((\sin x)^{-3} - x^{-3}\cos x)\,dx$,
(b) $\int_0^1 x^{-10} e^{-1/x}\,dx$.

4. Assume $f : [0, 1] \to \mathbb{R}$ is given by $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q} \end{cases}$. Compute $(U)\int_0^1 f$ and $(L)\int_0^1 f$. Does $\int_0^1 f$ exist?

5. If $f : [0, 1] \to \mathbb{R}$ is integrable, show that $g : [0, 1] \to \mathbb{R}$ given by $g(x) = \sqrt{x \arctan f(x) + 2}$ is integrable on $[0, 1]$. Then let

$$A = \left\{ \int_0^1 \sqrt{x \arctan f(x) + 2}\,dx : f : [0, 1] \to \mathbb{R} \text{ is integrable} \right\} \subset \mathbb{R}$$

Compute $\inf A$ and $\sup A$.

6. Compute:
(a) $\lim_{n\to\infty} \int_0^{\pi/2} \frac{n \sin x}{n+\sin x}\,dx$
(b) $\lim_{n\to\infty} \int_0^{1+1/n} \frac{nx^2+x^{2015}}{nx^3+x^2+4n}\,dx$.

7. True or false: if $f : [0, 1] \to \mathbb{R}$ is continuous, then $\int_0^\pi f(\sin x)\cos x\,dx = 0$.

8. For $x < 1$, let (a) $F(x) = \int_0^x (e^{t^2} - 1)\,dt$, (b) $F(x) = \int_0^x \frac{e^t-1}{1-t}\,dt$. Compute $\alpha = \lim_{x\to 0} x^{-2} F(x)$ and $\beta = \lim_{x\to 0} x^{-3}(F(x) - \alpha x^2)$. (*Advice*. It is much better to determine Taylor series for $F(x)$ up to x^3 instead of using L'Hôpital.)

Brief solutions

1. (a) No. $f(x) = |\sin(\pi x)|$ has the property that $f(n) = 0$ for every $n \in \mathbb{N}$. But it is a nonzero periodic function and so $\int_0^\infty f(x)\,dx = \infty$.

(b) No. Take the continuous function f such that, for all $n \in \mathbb{N}$, $f(n) = n$, f is linear on $[n, n+1/n^3]$, and on $[n-1/n^3, n]$, and 0 otherwise. Then $\int_{[n-1/n^3,n+1/n^3]} f = 1/n^2$. and so $\int_0^\infty f = \sum_{n=1}^\infty 1/n^2 < \infty$, but f is unbounded.

(c) If $\lim_{x\to\infty} f(x) = a$ exists as f is nonincreasing. If $a > 0$, then $f(x) \geq a$ for all x, and so $\int_0^\infty f \geq \int_0^\infty a = \infty$.

(d) Yes. If $|f'| \leq M$, then by the Mean Value Theorem $|f(x_2) - f(x_1)| = |f'(c)||x_2 - x_1| \leq M|x_2 - x_1|$ for $x_1, x_2 \geq 0$. Assume that is not true that $\lim_{x\to\infty} f(x) = 0$; then there exists an $\epsilon > 0$ and a sequence of $x_n \to \infty$, so that $f(x_n) \geq \epsilon$. We may assume that $\epsilon/M < 1$ and that x_n satisfy $x_{n+1} \geq x_n + 1$ for all $n \geq 1$. Let $I_n = [x_n - \epsilon/(2M), x_n + \epsilon/(2M)]$. The intervals I_n are pairwise disjoint. Also, as $f(x_n) \geq \epsilon$, $f(x) \geq \epsilon/2$ for $x \in I_n$. Thus

$$\int_0^\infty f(x)\,dx \geq \sum_{n=1}^\infty \int_{I_n} f(x)\,dx \geq \sum_{n=1}^\infty \frac{\epsilon}{2} \cdot \frac{1}{M} = \infty.$$

2. Let $f(x) = \frac{e^{\sin x} \sin x}{x^2+1}$. Then $|f(x)| \leq \frac{e}{x^2+1}$ and so $\int_0^\infty f(x)\,dx$ converges absolutely, with $\int_0^\infty |f(x)|\,dx \leq \frac{e\pi}{2}$. Therefore $\int_0^\infty f(x)\,dx$ converges.

3. (a) We use Taylor series. Let $f(x) = (\sin x)^{-3} - x^{-3}\cos x$ be the integrand. For $x < 1$, $\sin x = x - x^3/3 + \mathcal{O}(x^5)$, $\cos x = 1 - x^2/2 + \mathcal{O}(x^4)$, and $(1-x)^{-3} = 1 + 3x + \mathcal{O}(x^2)$. (Here, for example, $\mathcal{O}(x^5)$ is an expression whose absolute value divided by x^5 remains bounded as $x \to 0$; in our case, such expression is bounded on, say, $(0, 1/2]$ by Cx^5, for some constant C.)

$$\begin{aligned}
f(x) &= x^{-3}\left((1 - x^2/3 + \mathcal{O}(x^4))^{-3} - 1 + x^2/2 - \mathcal{O}(x^4)\right) \\
&= x^{-3}\left(1 + x^2 + \mathcal{O}(x^4) - 1 + x^2/2 + \mathcal{O}(x^4)\right) \\
&= x^{-3}\left(3x^2/2 + \mathcal{O}(x^4)\right) \\
&= x^{-1}\left(3/2 + \mathcal{O}(x^2)\right)
\end{aligned}$$

From this computation, there exists an $\delta > 0$ so that $f(x) \geq x^{-1}$ for $x < \delta$. But then $\int_0^\delta f(x)\,dx \geq \int_0^\delta x^{-1}\,dx = \infty$. Thus $\int_0^1 f(x)\,dx = \infty$. The integral diverges.

(b) By change of variable $z = 1/x$ (and then z is renamed to x), for small a,

$$\int_a^1 x^{-10} e^{-1/x}\,dx = \int_1^{1/a} e^{-x} x^8\,dx$$

and the last integral converges as $a \to 0$ Since $e^x \geq x^{10}$ for large x, $e^{-x}x^{-8} \leq x^{-2}$. As x^{-2} has a convergent integral on $[1, \infty)$, so does $e^{-x}x^{-8} \leq x^{-2}$. Thus the limit of the above expression as $a \to 0$ exists and the integral converges.

4. Let $g(x) = x$, $h(x) = x^2$. Observe that $g(x) \geq h(x)$ for $x \in [0, 1]$. As $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense, $U(f, P) = U(g, P)$ and $L(f, P) = U(h, P)$ for every partition P. Therefore, $(U)\int_0^1 f = (U)\int_0^1 g = \int_0^1 x\,dx = \frac{1}{2}$ and $(L)\int_0^1 f = (L)\int_0^1 h = \int_0^1 x^2\,dx = \frac{1}{3}$ and the integral does not exist.

5. As $\arctan x \in (-\pi/2, \pi/2)$ for every x,

$$\int_0^1 \sqrt{x \arctan f(x) + 2}\, dx \leq \int_0^1 \sqrt{x \cdot \pi/2 + 2}\, dx$$
$$= \frac{2}{\pi}\left((\pi/2+2)^{3/2} - 2^{3/2}\right).$$

If $f = M$ is a constant function for some large M, then

$$\int_0^1 \sqrt{x \arctan f(x) + 2}\, dx = \int_0^1 \sqrt{x \cdot \arctan M + 2}\, dx$$
$$= \frac{1}{\arctan M}\left((\arctan M + 2)^{3/2} - 2^{3/2}\right).$$

But $\arctan M$ converges to $\pi/2$ as $M \to \infty$, so

$$\sup A = \frac{2}{\pi}\left((\pi/2+2)^{3/2} - 2^{3/2}\right).$$

Similarly,

$$\inf A = \int_0^1 \sqrt{2 - x \cdot \pi/2}\, dx$$
$$= \frac{2}{\pi}\left(2^{3/2} - (2-\pi/2)^{3/2}\right)$$

6. (a) Let $f_n(x) = \frac{n \sin x}{n + \sin x}$ be the integrand function and $f(x) = \sin x$. Then

$$\sup_{x\in[0,\pi/2]} |f_n(x) - f(x)| = \sup_{x\in[0,\pi/2]} \frac{\sin^2 x}{n + \sin x} \leq \frac{1}{n}$$

This proves that the sequence of functions f_n converges to f uniformly. The limit therefore equals $\int_0^{\pi/2} \sin x\, dx = 1$.

(b) Let $f_n(x) = \frac{nx^2 + x^{2015}}{nx^3 + x^2 + 4n} = \frac{x^2 + x^{2015}/n}{x^3 + 4 + x^2/n}$ be the integrand function. Clearly $f_n \geq 0$.

If $x \leq 2$ and $n \geq 1$, we can easily bound $f_n(x) \leq 2^{2016}$, and so $0 \leq \int_1^{1+1/n} f_n(x)\, dx \leq 2^{2016}/n \to 0$. Thus we need to find the limit of $\int_0^1 f_n(x)\, dx$.

Let $f(x) = \frac{x^2}{x^3+4}$. For $x \in [0,1]$,

$$|f_n(x) - f(x)| = \frac{|(x^3+4)\cdot x^{2015}/n - x^4/n|}{(x^3+4+x^2/n)(x^3+4)} \leq \frac{6/n}{16} \leq \frac{1}{n}.$$

This proves that the sequence of functions f_n converges to f uniformly on $[0,1]$. The limit therefore equals $\int_0^1 f(x)\, dx = \frac{1}{3}(\log 5 - \log 4)$.

7. True. Use the change of variables with $g(x) = \sin x$, or directly the fundamental theorem of calculus with $F(x) = \int_0^x f$. Then the integral equals $F(\sin \pi) - F(\sin 0) = 0$.

8. (a) Since $e^{t^2} - 1 = t^2 + t^4/2 + \ldots$, with radius of convergence ∞, we have $F(x) = x^3/3 + x^5/10 + \ldots$, , with the same radius of convergence. Thus $\alpha = 0$ and $\beta = 1/3$.

(b) We have $e^t - 1 = t + t^2/2 + \ldots$ and $1/(1-t) = 1 + t + t^2 + \ldots$, first with radius of convergence ∞ and second with radius of convergence 1. Hence, for $|t| < 1$,

$$\frac{e^t - 1}{1-t} = t + \frac{3}{2}t^2 + \ldots$$

and so, for $|x| < 1$, $F(x) = x^2/2 + x^3/2 + \ldots$. Therefore $\alpha = 1/2$ and $\beta = 1/2$.