Discussion problems 5

Note. These are final problems on the Riemann integral, focusing particularly on the improper integral.

1. Assume $f : [0, \infty) \to [0, \infty)$ is continuous. For each statement below determine, with proof, whether it is true or false.
   (a) If $\sum_{n=1}^{\infty} f(n) < \infty$, then $\int_{0}^{\infty} f(x) \, dx$ converges.
   (b) If $\int_{0}^{\infty} f(x) \, dx$ converges, then $f$ is bounded on $[0, \infty)$.
   (c) If $\int_{0}^{\infty} f(x) \, dx$ converges and $f$ is nonincreasing (i.e., $0 \leq x_1 \leq x_2$ implies $f(x_1) \geq f(x_2)$), then $\lim_{x \to \infty} f(x) = 0$.
   (d) If $\int_{0}^{\infty} f(x) \, dx$ converges, $f$ is differentiable on $(0, \infty)$ and $f'$ is bounded on $(0, \infty)$, then $\lim_{x \to \infty} f(x) = 0$.

2. Prove that $\int_{0}^{\infty} e^{\sin x} \sin x \, dx$ converges.

3. For each integral below, determine whether it converges or not.
   (a) $\int_{0}^{1} \left( \sin x - \frac{3}{x} - x - 3 \cos x \right) \, dx$.
   (b) $\int_{0}^{1} x^{-10} e^{-1/x} \, dx$.

4. Assume $f : [0, 1] \to \mathbb{R}$ is given by $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q} \end{cases}$. Compute $(U) \int_{0}^{1} f$ and $(L) \int_{0}^{1} f$. Does $\int_{0}^{1} f$ exist?

5. If $f : [0, 1] \to \mathbb{R}$ is integrable, show that $g : [0, 1] \to \mathbb{R}$ given by $g(x) = \sqrt{x \arctan f(x) + 2}$ is integrable on $[0, 1]$. Then let

$$A = \left\{ \int_{0}^{1} \sqrt{x \arctan f(x) + 2} \, dx : f : [0, 1] \to \mathbb{R} \text{ is integrable} \right\} \subset \mathbb{R}$$

Compute $\inf A$ and $\sup A$.

6. Compute:
   (a) $\lim_{n \to \infty} \int_{0}^{\pi/2} \frac{n \sin x}{n + \sin x} \, dx$
   (b) $\lim_{n \to \infty} \int_{0}^{1+1/n} \frac{n x^2 + x^{2015}}{n x^3 + x^2 + 4n} \, dx$.

7. True or false: if $f : [0, 1] \to \mathbb{R}$ is continuous, then $\int_{0}^{1} f(x) \cos x \, dx = 0$.

8. For $x < 1$, let (a) $F(x) = \int_{0}^{x} (e^{t^2} - 1) \, dt$, (b) $F(x) = \int_{0}^{x} \frac{x^2 - 1}{t^3} \, dt$. Compute $\alpha = \lim_{x \to 0} x^{-2} F(x)$ and $\beta = \lim_{x \to 0} x^{-3} (F(x) - \alpha x^2)$. (Advice. It is much better to determine Taylor series for $F(x)$ up to $x^3$ instead of using L'Hôpital.)
1. (a) No. \( f(x) = |\sin(\pi x)| \) has the property that \( f(n) = 0 \) for every \( n \in \mathbb{N} \). But it is a nonzero periodic function and so \( \int_0^\infty f(x) \, dx = \infty \).

(b) No. Take the continuous function \( f \) such that, for all \( n \in \mathbb{N} \), \( f(n) = n \), \( f \) is linear on \([n, n+1/3^n]\), and on \([n-1/3^n, n]\), and 0 otherwise. Then \( \int_{n-1/3^n}^{n+1/3^n} f = 1/n^2 \), so that \( \sum_{n=1}^\infty 1/n^2 < \infty \), and \( f \) is unbounded.

(c) If \( \lim_{x \to \infty} f(x) = a \) exists as \( f \) is nonincreasing. If \( a > 0 \), then \( f(x) \geq a \) for all \( x \), and so \( \int_0^\infty f(x) \, dx \geq \sum_{n=1}^\infty \int_{I_n} f(x) \, dx \geq \sum_{n=1}^\infty \frac{\varepsilon}{2} \frac{1}{M} = \infty \).

(d) Yes. If \( |f'| \leq M \), then by the Mean Value Theorem \( |f(x_2) - f(x_1)| = |f'(c)||x_2 - x_1| \leq M|x_2 - x_1| \) for \( x_1, x_2 \geq 0 \). Assume that is not true that \( \lim_{x \to \infty} f(x) = 0 \); then there exists an \( \varepsilon > 0 \) and a sequence of \( x_n \to \infty \), so that \( f(x_n) \geq \varepsilon \). We may assume that \( \varepsilon/M < 1 \) and that \( x_n \) satisfy \( x_{n+1} \geq x_n + 1 \) for all \( n \geq 1 \). Let \( I_n = [x_n - \varepsilon/(2M), x_n + \varepsilon/(2M)] \). The intervals \( I_n \) are pairwise disjoint. Also, as \( f(x_n) \geq \varepsilon \), \( f(x) \geq \varepsilon/2 \) for \( x \in I_n \). Thus
\[
\int_0^\infty f(x) \, dx \geq \sum_{n=1}^\infty \int_{I_n} f(x) \, dx \geq \sum_{n=1}^\infty \frac{\varepsilon}{2} \frac{1}{M} = \infty.
\]

2. Let \( f(x) = \frac{\sin x \sin x}{x^2 + 1} \). Then \( |f(x)| \leq \frac{e}{x^2 + 1} \) and so \( \int_0^\infty f(x) \, dx \) converges absolutely, with \( \int_0^\infty |f(x)| \, dx \leq \frac{e}{2} \). Therefore \( \int_0^\infty f(x) \, dx \) converges.

3. (a) We use Taylor series. Let \( f(x) = (\sin x)^{-3} - x^{-3} \cos x \) be the integrand. For \( x < 1 \), \( \sin x = x - x^3/3 + O(x^5) \), \( \cos x = 1 - x^2/2 + O(x^4) \), and \( (1 - x)^{-3} = 1 + 3x + O(x^2) \). (Here, for example, \( O(x^5) \) is an expression whose absolute value divided by \( x^5 \) remains bounded as \( x \to 0 \); in our case, such expression is bounded on, say, \((0, 1/2]\) by \( Cx^3 \), for some constant \( C \).)
\[
f(x) = x^{-3} ((1 - x^2/3 + O(x^4))^{-3} - 1 + x^2/2 - O(x^4))
\]
\[
= x^{-3} (1 + x^2 + O(x^4) - 1 + x^2/2 + O(x^4))
\]
\[
= x^{-3} (3x^2/2 + O(x^4))
\]
\[
= x^{-1} (3/2 + O(x^2))
\]

From this computation, there exists an \( \delta > 0 \) so that \( f(x) \geq x^{-1} \) for \( x < \delta \). But then \( \int_0^\delta f(x) \, dx \geq \int_0^\delta x^{-1} \, dx = \infty \). Thus \( \int_0^1 f(x) \, dx = \infty \). The integral diverges.

(b) By change of variable \( z = 1/x \) (and then \( z \) is renamed to \( x \)), for small \( a \),
\[
\int_a^1 x^{-10}e^{-1/x} \, dx = \int_1^{1/a} e^{-x}x^8 \, dx
\]
and the last integral converges as \( a \to 0 \) Since \( e^x \geq x^{10} \) for large \( x \), \( e^{-x}x^{-8} \leq x^{-2} \). As \( x^{-2} \) has a convergent integral on \([1, \infty)\), so does \( e^{-x}x^{-8} \leq x^{-2} \). Thus the limit of the above expression as \( a \to 0 \) exists and the integral converges.

4. Let \( g(x) = x, h(x) = x^2 \). Observe that \( g(x) \geq h(x) \) for \( x \in [0, 1] \). As \( \mathbb{Q} \) and \( \mathbb{R} \setminus \mathbb{Q} \) are both dense, \( U(f, P) = U(g, P) \) and \( L(f, P) = U(h, P) \) for every partition \( P \). Therefore, \( (U) \int_0^1 f = (U) \int_0^1 g = \int_0^1 x \, dx = \frac{1}{2} \) and \( (L) \int_0^1 f = (L) \int_0^1 h = \int_0^1 x^2 \, dx = \frac{1}{3} \) and the integral does not exist.
5. As \( \arctan x \in (-\pi/2, \pi/2) \) for every \( x \),
\[
\int_0^1 \sqrt{x} \arctan f(x) + 2 \, dx \leq \int_0^1 \sqrt{x} \cdot \pi/2 + 2 \, dx = \frac{2}{\pi} \left( (\pi/2 + 2)^{3/2} - 2^{3/2} \right).
\]
If \( f = M \) is a constant function for some large \( M \), then
\[
\int_0^1 \sqrt{x} \arctan f(x) + 2 \, dx = \int_0^1 \sqrt{x} \cdot \arctan M + 2 \, dx = \frac{1}{\arctan M} \left( (\arctan M + 2)^{3/2} - 2^{3/2} \right).
\]
But \( \arctan M \) converges to \( \pi/2 \) as \( M \to \infty \), so
\[
\sup A = \frac{2}{\pi} \left( (\pi/2 + 2)^{3/2} - 2^{3/2} \right).
\]
Similarly,
\[
\inf A = \int_0^1 \sqrt{2 - x} \cdot \pi/2 \, dx = \frac{2}{\pi} \left( 2^{3/2} - (2 - \pi/2)^{3/2} \right).
\]

6. (a) Let \( f_n(x) = \frac{n \sin x}{n + \sin x} \) be the integrand function and \( f(x) = \sin x \). Then
\[
\sup_{x \in [0, \pi/2]} |f_n(x) - f(x)| = \sup_{x \in [0, \pi/2]} \frac{\sin^2 x}{n + \sin x} \leq \frac{1}{n}
\]
This proves that the sequence of functions \( f_n \) converges to \( f \) uniformly. The limit therefore equals
\[
\int_0^{\pi/2} \sin x \, dx = 1.
\]
(b) Let \( f_n(x) = \frac{n x^2 + x^{2015}}{n x^4 + x^2 + 4n} = \frac{x^2 + x^{2015}/n}{x^3 + 4 + x^2/n} \) be the integrand function. Clearly \( f_n \geq 0 \).

If \( x \leq 2 \) and \( n \geq 1 \), we can easily bound \( f_n(x) \leq 2^{2016} \), and so
\[
0 \leq \int_{-1}^{1 + 1/n} f_n(x) \, dx \leq 2^{2016}/n \to 0.
\]
Thus we need to find the limit of \( \int_0^1 f_n(x) \, dx \).
Let \( f(x) = \frac{x^2}{x^3 + 4} \). For \( x \in [0, 1] \),
\[
|f_n(x) - f(x)| = \left| \frac{(x^3 + 4) \cdot x^{2015}/n - x^4/n}{(x^3 + 4 + x^2/n)(x^3 + 4)} \right| \leq \frac{6/n}{16} \leq \frac{1}{n}
\]
This proves that the sequence of functions \( f_n \) converges to \( f \) uniformly on \([0, 1] \). The limit therefore equals
\[
\int_0^1 f(x) \, dx = \frac{1}{3}(\log 5 - \log 4).
\]

7. True. Use the change of variables with \( g(x) = \sin x \), or directly the fundamental theorem of calculus with \( F(x) = \int_0^x f \). Then the integral equals \( F(\pi) - F(0) = 0 \).

8. (a) Since \( e^{t^2} = 1 + t^2/2 + \ldots \), with radius of convergence \( \infty \), we have \( F(x) = x^3/3 + x^5/10 + \ldots \), with the same radius of convergence. Thus \( \alpha = 0 \) and \( \beta = 1/3 \).
(b) We have \( e^t - 1 = t + t^2/2 + \ldots \) and \( 1/(1-t) = 1 + t + t^2 + \ldots \), first with radius of convergence \( \infty \) and second with radius of convergence \( 1 \). Hence, for \( |t| < 1 \),
\[
\frac{e^t - 1}{1 - t} = t + \frac{3}{2}t^2 + \ldots
\]
and so, for \( |x| < 1 \), \( F(x) = x^2/2 + x^3/2 + \ldots \). Therefore \( \alpha = 1/2 \) and \( \beta = 1/2 \).