Math 125B, Winter 2015.

Discussion problems 6

Note. These are problems on topology of $\mathbb{R}^n$.

1. Show that $[0,1]\times[0,1]$ includes a countable dense set S such that no two points of S lie on the same vertical line or on the same horizontal line.

2. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined as follows:

$$f(x,y)=\begin{cases}\frac{|x|^\alpha|y|^\beta}{(x^2+xy+y^2)(|x|+|y|)} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Show that f is indeed defined on the whole of $\mathbb{R}^2$. Determine for which values of parameters $\alpha,\beta\geq 0$ is f continuous on $\mathbb{R}^2$.

3. Let $f:\mathbb{R}^3\to\mathbb{R}$ be defined as follows:

$$f(x,y,z)=\begin{cases}\frac{xy|z|^\alpha}{|x|^{3-\alpha}+y^2+z^2} & (x,y,z)\neq(0,0,0)\\ 0 & (x,y,z)=(0,0,0)\end{cases}$$

Determine for which values of parameter $\alpha\in[0,3]$ is f continuous on $\mathbb{R}^3$.

4. Let $T:\mathbb{R}^n\to\mathbb{R}^m$ be a linear map. (a) Show that there exists a constant M so that $||T(x)||\leq M||x||$. (b) Show that T is uniformly continuous.

5. Assume $f:\mathbb{R}^2\to\mathbb{R}$. True or false: if $\lim_{r\to 0} f(ra,rb)=0$ for every $a,b\in\mathbb{R}$ with $a^2+b^2=1$, then $\lim_{(x,y)\to(0,0)} f(x,y)=0$.

6. Assume $f:\mathbb{R}^n\to\mathbb{R}^2$, let f_1 and f_2 be its coordinate functions, and fix $a\in\mathbb{R}^n$. True or false: if $\lim_{x\to a}(f_1(x)^4+|f_2(x)|)=0$, then $\lim_{x\to a} f(x)=0$.

Brief solutions

1. We construct $S_1 \subset S_2 \subset S_3 \subset \dots$ by induction. We will denote by $X_k \subset [0,1]$ and $Y_k \subset [0,1]$ the sets of first and second coordinates of points in S_k.

Let S_1 consist of an arbitrary point in $[0,1] \times [0,1]$. For each $k \geq 1$, find a square L of the largest size which does not include a point in S_k (choosing arbitrarily if there is more than one such square of the largest size). Assume $L = [a,b] \times [c,d]$. As $X_k \cap [a,b]$ and $Y_k \cap [c,d]$ are finite, there exist rational numbers $p \in [a,b] \setminus X_k$ and $q \in [c,d] \setminus Y_k$. Then $(p,q) \in L$, but the horizontal and vertical lines through (p,q) include no point in S_k. Define $S_{k+1} = S_k \cup \{(p,q)\}$. In this fashion, the set $S = \cup_k S_k$ contains at least one point in every square.

Clearly S is countable. To show that S is dense, observe that for any $x \in [0,1] \times [0,1]$ there is a sequence of squares L_k, which include x and with sizes going to 0. Each of them includes a point $x_k \in L_k \cap S$. Thus $x_k \to x$.

2. To show that f is defined everywhere, observe that

$$x^2 + xy + y^2 = \frac{1}{2}(x+y)^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2,$$

and so $x^2 + xy + y^2 > 0$ unless $x = y = 0$. We only need to prove continuity at $(0,0)$. Write $(x,y) = r(a,b)$, where $a^2 + b^2 = 1$. Then $|a|, |b| \leq 1$ and one of $|a|$, $|b|$ is at least $1/\sqrt{2} > 1/2$. Thus

$$f(x,y) = r^{\alpha+\beta-3} \frac{|a|^\alpha |b|^\beta}{(a^2+ab+b^2)(|a|+|b|)} \leq r^{\alpha+\beta-3} \frac{1}{\frac{1}{4} \cdot \frac{1}{2}} = 8r^{\alpha+\beta-3}.$$

This shows that f is continuous if $\alpha + \beta - 3 > 0$. If $\alpha + \beta - 3 = 0$, then f is a different constant on different straight lines through the origin, so f cannot be continuous. If $\alpha + \beta - 3 < 0$, and, say, $a = b = 1/\sqrt{2}$, then $f(x,y) \to \infty$ as $r \to 0$, so f again cannot be continuous. The answer is that f is continuous exactly when $\alpha + \beta > 3$.

3. Again, only continuity at the origin is an issue. Write $x = r(a,b,c)$ where $a^2 + b^2 + c^2 = 1$. Then all $|a|$, $|b|$, $|c|$ are at most 1, and one of them is at least $1/\sqrt{3} \geq 1/2$. Furthermore,

$$|f(x,y,z)| = r^{\alpha+2} \frac{|a||b||c|^\alpha}{r^{3-\alpha}|a|^{3-\alpha} + r^2(b^2+c^2)}$$

If $\alpha \in (0,1]$, then $3 - \alpha \geq 2$ and so for $r \leq 1$,

$$|f(x,y,z)| \leq r^\alpha \frac{|a||b||c|^\alpha}{|a|^{3-\alpha} + b^2 + c^2} \leq r^\alpha \frac{1}{2^{-3+\alpha}} \to 0$$

as $r \to 0$. If $\alpha = 0$ and (a,b,c) constant, with $a \neq 1$,

$$f(x,y,z) \to \frac{ab}{b^2+c^2}$$

so that the limit of f as $(x,y,z) \to 0$ does not exist. If $\alpha \in [1,3]$, then for $r \leq 1$,

$$|f(x,y,z)| \leq r^{2\alpha-1} \frac{|a||b||c|^\alpha}{|a|^{3-\alpha} + b^2 + c^2} \leq 4r \to 0.$$

The function is continuous unless $\alpha = 0$.

4. For (a), it suffices to show this for the norm $||\cdot||_\infty$ instead of the Euclidean one. Let T be given in the standard basis by the $m \times n$ matrix $[t_{ij}]$, and let $M = n \cdot \max_{i,j} |t_{ij}|$. Then

$$||T(x)||_\infty = \max_{i=1,\ldots,m} |\sum_{j=1}^{n} t_{ij}x_j| \le \max_{i=1,\ldots,m} \sum_{j=1}^{n} |t_{ij}| \cdot ||x||_\infty \le M||x||_\infty.$$

(In fact, one can show that, for the Euclidean norm, the smallest M is the largest singular value of M, a.k.a. the largest eigenvalue of M^TM.) For (b), fix an $\epsilon > 0$ and let $\delta = \epsilon/M$. Assuming $||x_1 - x_2|| < \delta$, we get $||T(x_1) - T(x_2)|| = ||T(x_1 - x_2)|| \le M||x_1 - x_2|| < \epsilon$.

5. No. Let

$$f(x,y) = \begin{cases} 1 & y = x^2 \\ 0 & \text{otherwise} \end{cases}$$

Assume that (x, y) is on a straight line $(x, y) = r(a, b)$, $r > 0$. If $b = 0$ (and thus $a \neq 0$), $x^2 - y = a^2r^2 > 0$ for all $r > 0$. If $b > 0$, then $x^2 - y = r(ra^2 - b) < 0$ for $r < b/a^2$. If $b < 0$, then $x^2 - y \ge -br > 0$ for all $r > 0$. In all cases, $f(ra, rb) = 0$ for small enough r and thus $\lim_{r\to 0} f(ra, rb) = 0$. However, f clearly does not have a limit as $(x, y) \to 0$.

6. Yes. Under the assumption, $f_1(x) \to 0$ and $f_2(x) \to 0$ as $x \to a$, and then $f(x) \to 0$.