Discussion problems 6

Note. These are problems on topology of $\mathbb{R}^n$.

1. Show that $[0,1] \times [0,1]$ includes a countable dense set $S$ such that no two points of $S$ lie on the same vertical line or on the same horizontal line.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as follows:

$$f(x, y) = \begin{cases} \frac{|x|^3 |y|^3}{(x^2 + xy + y^2)(|x| + |y|)} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Show that $f$ is indeed defined on the whole of $\mathbb{R}^2$. Determine for which values of parameters $\alpha, \beta \geq 0$ is $f$ continuous on $\mathbb{R}^2$.

3. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined as follows:

$$f(x, y, z) = \begin{cases} \frac{xy|z|^3}{|x| + |y| + z^2} & (x, y, z) \neq (0, 0, 0) \\ 0 & (x, y, z) = (0, 0, 0) \end{cases}$$

Determine for which values of parameter $\alpha \in [0, 3]$ is $f$ continuous on $\mathbb{R}^3$.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. (a) Show that there exists a constant $M$ so that $||T(x)|| \leq M||x||$.

(b) Show that $T$ is uniformly continuous.

5. Assume $f : \mathbb{R}^2 \to \mathbb{R}$. True or false: if $\lim_{r \to 0} f(ra, rb) = 0$ for every $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$, then $\lim_{(x,y) \to (0,0)} f(x, y) = 0$.

6. Assume $f : \mathbb{R}^n \to \mathbb{R}^2$, let $f_1$ and $f_2$ be its coordinate functions, and fix $a \in \mathbb{R}^n$. True or false: if $\lim_{x \to a}(f_1(x)^4 + |f_2(x)|) = 0$, then $\lim_{x \to a} f(x) = 0$. 


Brief solutions

1. We construct \( S_1 \subset S_2 \subset S_3 \subset \ldots \) by induction. We will denote by \( X_k \subset [0,1] \) and \( Y_k \subset [0,1] \) the sets of first and second coordinates of points in \( S_k \).

   Let \( S_1 \) consist of an arbitrary point in \([0,1] \times [0,1] \). For each \( k \geq 1 \), find a square \( L \) of the largest size which does not include a point in \( S_k \) (choosing arbitrarily if there is more than one such square of the largest size). Assume \( L = [a, b] \times [c, d] \). As \( X_k \cap [a, b] \) and \( Y_k \cap [c, d] \) are finite, there exist rational numbers \( p \in [a, b] \setminus X_k \) and \( q \in [c, d] \setminus Y_k \). Then \( (p, q) \in L \), but the horizontal and vertical lines through \((p, q)\) include no point in \( S_k \). Define \( S_{k+1} = S_k \cup \{(p, q)\} \). In this fashion, the set \( S = \cup_k S_k \) contains at least one point in every square.

   Clearly \( S \) is countable. To show that \( S \) is dense, observe that for any \( x \in [0,1] \times [0,1] \) there is a sequence of squares \( L_k \), which include \( x \) and with sizes going to 0. Each of them includes a point \( x_k \in L_k \cap S \). Thus \( x_k \to x \).

2. To show that \( f \) is defined everywhere, observe that

   \[
   x^2 + xy + y^2 = \frac{1}{2}(x+y)^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2,
   \]

   and so \( x^2 + xy + y^2 > 0 \) unless \( x = y = 0 \). We only need to prove continuity at \((0,0)\). Write \((x,y) = r(a,b)\), where \( a^2 + b^2 = 1 \). Then \( |a|, |b| \leq 1 \) and one of \( |a|, |b| \) is at least \( 1/\sqrt{2} > 1/2 \). Thus

   \[
   f(x,y) = r^{\alpha + \beta - 3} \frac{|a| |b|^\beta}{(a^2 + ab + b^2)(|a| + |b|)} \leq r^{\alpha + \beta - 3} \frac{1}{\frac{1}{4} \cdot \frac{1}{2}} = 8r^{\alpha + \beta - 3}.
   \]

   This shows that \( f \) is continuous if \( \alpha + \beta - 3 > 0 \). If \( \alpha + \beta - 3 = 0 \), then \( f \) is a different constant on different straight lines through the origin, so \( f \) cannot be continuous. If \( \alpha + \beta - 3 < 0 \), and, say, \( a = b = 1/\sqrt{2} \), then \( f(x,y) \to \infty \) as \( r \to 0 \), so \( f \) again cannot be continuous. The answer is that \( f \) is continuous exactly when \( \alpha + \beta > 3 \).

3. Again, only continuity at the origin is an issue. Write \( x = r(a,b,c) \) where \( a^2 + b^2 + c^2 = 1 \). Then all \( |a|, |b|, |c| \) are at most 1, and one of them is at least \( 1/\sqrt{3} \geq 1/2 \). Furthermore,

   \[
   |f(x,y,z)| = r^{\alpha + 2} \frac{|a||b||c|^\alpha}{r^{3-\alpha} |a|^{3-\alpha} + r^2(b^2 + c^2)}
   \]

   If \( \alpha \in (0,1] \), then \( 3 - \alpha \geq 2 \) and so for \( r \leq 1 \),

   \[
   |f(x,y,z)| \leq r^\alpha \frac{|a||b||c|^\alpha}{|a|^{3-\alpha} + b^2 + c^2} \leq r^\alpha \frac{1}{2^{-3+\alpha}} \to 0
   \]

   as \( r \to 0 \). If \( \alpha = 0 \) and \((a,b,c)\) constant, with \( a \neq 1 \),

   \[
   f(x,y,z) \to \frac{ab}{b^2 + c^2}
   \]

   so that the limit of \( f \) as \((x,y,z) \to 0\) does not exist. If \( \alpha \in [1,3) \), then for \( r \leq 1 \),

   \[
   |f(x,y,z)| \leq r^{2\alpha - 1} \frac{|a||b||c|^\alpha}{|a|^{3-\alpha} + b^2 + c^2} \leq 4r \to 0.
   \]

   The function is continuous unless \( \alpha = 0 \).
4. For (a), it suffices to show this for the norm $|| \cdot ||_\infty$ instead of the Euclidean one. Let $T$ be given in the standard basis by the $m \times n$ matrix $[t_{ij}]$, and let $M = n \cdot \max_{i,j} |t_{ij}|$. Then

$$||T(x)||_\infty = \max_{i=1,...,m} \left| \sum_{j=1}^{n} t_{ij} x_j \right| \leq \max_{i=1,...,m} \sum_{j=1}^{n} |t_{ij}| \cdot ||x||_\infty \leq M ||x||_\infty.$$ 

(In fact, one can show that, for the Euclidean norm, the smallest $M$ is the largest singular value of $M$, a.k.a. the largest eigenvalue of $M^T M$.) For (b), fix an $\epsilon > 0$ and let $\delta = \epsilon / M$. Assuming $||x_1 - x_2|| < \delta$, we get $||T(x_1) - T(x_2)|| = ||T(x_1 - x_2)|| \leq M ||x_1 - x_2|| < \epsilon$.

5. No. Let 

$$f(x, y) = \begin{cases} 1 & y = x^2 \\ 0 & \text{otherwise} \end{cases}$$

Assume that $(x, y)$ is on a straight line $(x, y) = r(a, b), r > 0$. If $b = 0$ (and thus $a \neq 0$), $x^2 - y = a^2 r^2 > 0$ for all $r > 0$. If $b > 0$, then $x^2 - y = r(a^2 - b) < 0$ for $r < b/a^2$. If $b < 0$, then $x^2 - y \geq -br > 0$ for all $r > 0$. In all cases, $f(ra, rb) = 0$ for small enough $r$ and thus $\lim_{r \to 0} f(ra, rb) = 0$. However, $f$ clearly does not have a limit as $(x, y) \to 0$.

6. Yes. Under the assumption, $f_1(x) \to 0$ and $f_2(x) \to 0$ as $x \to a$, and then $f(x) \to 0$. 