Math 125B, Winter 2015.

Discussion problems 7

Note. These are problems on partial derivatives.

1. The function $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) \neq (0,0) \end{cases}$$

(a) Is f in $\mathcal{C}(\mathbb{R}^2)$? In $\mathcal{C}^1(\mathbb{R}^2)$? In $\mathcal{C}^2(\mathbb{R}^2)$?

(b) Do $\frac{\partial^2}{\partial x \partial y} f$ and $\frac{\partial^2}{\partial y \partial x} f$? exist? Are they equal? Discuss the implications, if any, of the relevant theorem we proved in class.

2. Assume that $A \subset \mathbb{R}^2$ is open and $f : A \to \mathbb{R}$ is uniformly continuous, and that a rectangle $[a,b] \times [c,d] \subset A$.

(a) Show that $g(x) = \int_c^d f(x,y)\,dy$ and $h(y) = \int_a^b f(x,y)\,dx$ exist and are uniformly continuous on $[a,b]$ and $[c,d]$, respectively.

(b) Assume that $\frac{\partial f}{\partial y}$ is uniformly continuous on A, and let $f_k(x,y) = (f(x,y+k) - f(x,y))/k$. Show that, as $k \to 0$, $f_k \to \frac{\partial f}{\partial y}$, uniformly on A.

(c) Under the assumption in (b), show that $h'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$

(d) By differentiating both sides with respect to b, prove that $\int_a^b g(x)\,dx = \int_c^d h(y)\,dy$.

(*Remark*. This of course says that the order of integration in a double integral can be reversed. Try not to even write down double integrals in the proof, as they are bound to be confusing the first time you encounter them. Exchanging order of integration holds in great generality and is known as Fubini's Theorem.)

Brief solutions

1. We solve (a) and (b) together. Continuity of f at $(0,0)$ is clear. (Using the polar representation $(x,y) = r(a,b)$, with $a^2+b^2=1$, one gets that $f(x,y)$ equals r^2 times a bounded factor.) Therefore, $f \in \mathcal{C}(\mathbb{R}^2)$. We have, for $(x,y) \neq (0,0)$,

$$\frac{\partial f}{\partial x} = \frac{2xy^2}{x^2+y^2} - \frac{2x^3y^2}{(x^2+y^2)^2},$$

and $\frac{\partial f}{\partial x}(0,0) = 0$, from which you can clearly see that $\frac{\partial f}{\partial x}$ is continuous on $\mathbb{R}^2$. By symmetry $\frac{\partial f}{\partial y}$ is also continuous on $\mathbb{R}^2$, and so $f \in \mathcal{C}^1(\mathbb{R}^2)$. Moreover $\frac{\partial f}{\partial x}(0,y) = 0$ for every $x \neq 0$, and $\frac{\partial f}{\partial y}(x,0) = 0$ for every $y \neq 0$, thus $\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(0,0) = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}(0,0) = 0$. As the mixed partial derivatives are equal at any other point (by the fact that $f \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0.0)\})$), $\frac{\partial}{\partial y}\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial x}\frac{\partial f}{\partial y}$ exist and are equal on $\mathbb{R}^2$.

However, $f \notin \mathcal{C}^2(\mathbb{R}^2)$; for example

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}(x,y) = \frac{8x^3y^3}{(x^2+y^2)^3}$$

does not have a limit as $(x,y) \to (0,0)$. The two mixed derivatives can be equal even though neither is continuous; that is, continuity is sufficient but not necessary for equality of mixed derivatives.

2.(a) For any fixed x, the function $y \mapsto f(x,y)$ is continuous, as is the function $x \mapsto f(x,y)$ for a fixed y. Thus the integrals exist.

To show that g is uniformly continuous, pick an $\epsilon > 0$. Choose a $\delta > 0$ so that $||(x,y)-(x',y')|| < \delta$ implies $|f(x,y) - f(x',y')| < \epsilon/(|b-a|+|d-c|+1)$. Assume $|x-x'| < \delta$; then $||(x,y)-(x',y)|| < \delta$ and so $|g(x') - g(x)| \leq (b-a)\cdot\epsilon/((b-a)+(d-c)+1) < \epsilon$. A similar argument demonstrates uniform continuity of h.

(b) By the mean value theorem $f_k(x,y) = \frac{\partial f}{\partial y}(x,t)$, for some $t \in [y, y+k]$ that depends on x and y.

For a fixed $\epsilon > 0$, choose a $\delta > 0$ so that $||(x,y)-(x',y')|| < \delta$ implies $\left|\frac{\partial f}{\partial y}(x,y) - \frac{\partial f}{\partial y}(x',y')\right| < \epsilon$. Assume $k < \delta$. Then $||(x,y)-(x,t)|| < \delta$ and so

$$\left|f_k(x,y) - \frac{\partial f}{\partial y}(x,y)\right| = \left|\frac{\partial f}{\partial y}(x,t) - \frac{\partial f}{\partial y}(x,y)\right| < \epsilon$$

for all $(x,y) \in A$, proving uniform convergence.

(c) By linearity

$$\frac{h(y+k)-h(y)}{k} = \int_a^b f_k(x,y)\,dx$$

and then by uniform convergence proved in (b),

$$\int_a^b f_k(x,y)\,dx \to \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

as $k \to 0$ (uniformly in y, in fact). The proves, by definition of the derivative, that $h'(y)$ exists, and that the claimed equality holds.

(d) Keeping a fixed, let $F(z,y) = \int_a^z f(x,y)\,dx$, $G(z) = \int_a^z g(x)\,dx$ and $H(z) = \int_c^d F(z,y)\,dy$. These are uniformly continuous functions. To prove this for F, assume that $|f| \leq M$ on the square and pick an $\epsilon > 0$. Assume that $\delta > 0$ is such that $||(x,y)-(x',y')|| < \delta$ implies $|f(x,y)-f(x',y')| < \epsilon/(2(b-a))$. Now assume $||(z,y)-(z',y')|| < \min(\delta, \epsilon/(2M))$. Then

$$\begin{aligned}
|F(z',y') - F(z,y)| &= \left| \int_{z'}^{z} f(x,y')\,dx + \int_{a}^{z} (f(x,y') - f(x,y))\,dx \right| \\
&\leq \int_{z'}^{z} |f(x,y')|\,dx + \int_{a}^{z} \big|f(x,y') - f(x,y))\big|\,dx \\
&\leq M|z'-z| + (b-a)\cdot \epsilon/(b-a) < \epsilon,
\end{aligned}$$

as $|z-z'| < \epsilon/(2M)$ and $||(x,y')-(x,y)|| = |y'-y| < \delta$. By the fundamental theorem of calculus,

$$\frac{\partial F}{\partial z}(z,y) = f(z,y),$$

which is by the assumption uniformly continuous. Again, by the fundamental theorem of calculus

$$\frac{d}{dz}G(z) = g(z),$$

and by (c)

$$\frac{d}{dz}\int_c^d F(z,y)\,dy = \int_c^d \frac{\partial F}{\partial z}(z,y)\,dy = \int_c^d f(z,y)\,dy = g(z).$$

As $G(a) = H(a) = 0$, it follows that $G(z) = H(z)$ for all $z \in [a,b]$; in particular $G(b) = H(b)$.