
Discussion problems 8

Note. These are problems on differentiable functions.

1. Assume \( f : \mathbb{R}^2 \to \mathbb{R} \) is a function. For each statement below determine, with proof, whether it is true or false.
   (a) If \( f \in C^1(\mathbb{R}^2) \), then \( f \) is differentiable on \( \mathbb{R}^2 \).
   (b) If \( f \) is differentiable on \( \mathbb{R}^2 \), then \( f \in C^1(\mathbb{R}^2) \).
   (c) If \( f \) is differentiable on \( \mathbb{R}^2 \), and \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) is linear, then the function \( g : \mathbb{R}^2 \to \mathbb{R}^2 \), given by \( g(x) = (\sqrt{f(x)^2 + 1}, f(Tx)) \) is differentiable.
   (d) If \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) both exist and are continuous on \( \mathbb{R}^2 \), and \( \frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0 \), then the directional derivative \( D_u f(0,0) = 0 \) for every \( u \neq (0,0) \).

2. Let \( f : \mathbb{R}^3 \to \mathbb{R}^3 \) be given by \( f(x_1, x_2, x_3) = (x_1e^{x_2+2x_3}, 2x_2 \cos(x_1x_3), x_3e^{4x_1+3x_2}) \).
   (a) Compute \( Df(0,0,0) \).
   (b) Compute \( D(f \circ f)(0,0,0) \). (After solving (a), ten seconds, no computer.)
   (c) Is \( \frac{\partial^2}{\partial x \partial y} f(0,0) = \frac{\partial^2}{\partial y \partial x} f(0,0) \)? (Ten seconds, no computer)

3. Discuss differentiability of \( f : \mathbb{R}^2 \to \mathbb{R} \) defined as follows:
   \[
   f(x, y) = \begin{cases} 
   \frac{x^3-y^3}{x^2+y^2} & (x, y) \neq (0,0) \\
   0 & (x, y) = (0,0)
   \end{cases}
   \]
   (Remark. You are encouraged to use the computer for the tedious computing of partial derivatives. If the problem were on an exam, it would include something like the following: “You may assume that \( f_x = \frac{x(x^2+3y^2+2y^3)}{(x^2+y^2)^2} \) wherever applicable. Show that \( f(x, y) = -f(y, x) \) implies that \( f_y(x, y) = -f_x(y, x) \).”)

4. Discuss differentiability, in dependence of the parameter \( \alpha \in \mathbb{R} \), of \( f : \mathbb{R}^3 \to \mathbb{R} \) defined as follows:
   \[
   f(x, y, z) = \begin{cases} 
   \frac{xyz}{(x^2+y^2+z^2)^{\alpha}} & (x, y, z) \neq (0,0,0) \\
   0 & (x, y, z) = (0,0,0)
   \end{cases}
   \]
Brief solutions

1. (a) Yes. This follows from a theorem proved in class.
(b) No. We gave a counterexample in the lecture.
(c) Yes. Both coordinate functions of $g$ are differentiable because of the chain rule. The first coordinate function $g_1 : \mathbb{R}^2 \to \mathbb{R}$ is given by $g_1 = h \circ f$, where $h : \mathbb{R} \to \mathbb{R}$ is the function $h(x) = \sqrt{x^2 + 1}$ and is differentiable on $\mathbb{R}$. The second coordinate function $g_2 : \mathbb{R}^2 \to \mathbb{R}$ is given by $g_2 = f \circ T$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and therefore differentiable.
(d) Yes. Under the assumption, $Df(0,0)$ exists and equals $T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $D_u(f)(0,0) = Tu = 0$.

2. (a) By computing partial derivatives
$$Df(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
(b) By the chain rule, the answer is the square of the matrix in (a), that is
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
(c) Yes, this function is in $C^\infty(\mathbb{R}^2)$, so the two mixed derivatives are equal.

3. Clearly, $f \in C^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$ and so $f$ is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$. Also, it is easy to check by polar representation that $f \in C(\mathbb{R}^2)$. Moreover, $f(x,0) = x$ whenever $x \neq 0$, so
$$f_x(x,y) = \begin{cases} \frac{x(x^3+3xy^2+2y^3)}{(x^2+y^2)^{3/2}} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$
As $f_x(0,y) = 0$ for every $y \neq 0$, $f_x$ is clearly not continuous, so we need to check differentiability directly.

We have $f(0,y) = -y$ whenever $y \neq 0$, so $f_y(0,0) = -1$. Thus, the only candidate for $Df(0,0)$ is $\begin{bmatrix} 1 & -1 \end{bmatrix}$, and we need to check whether
$$0 = \lim_{(x,y) \to (0,0)} \frac{f(x,y) - x + y}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{yx^2 - xy^2}{(x^2 + y^2)^{3/2}}.$$ 
If $(x,y) = r(a,b)$, with $a^2 + b^2 = 1$, then the above expression is $ba^2 - ab^2$ and so the limit does not exist. Therefore $f$ is not differentiable at $(0,0)$.

4. It is easy to check by polar representation that $f$ is continuous exactly when $\alpha < 3/2$. Moreover, for all $\alpha$,
$$f_x(x,y,z) = \begin{cases} \frac{yz(x^2 - 2\alpha(x^2 + y^2 + z^2))}{(x^2 + y^2 + z^2)^{\alpha+1}} & (x,y,z) \neq (0,0,0) \\ 0 & (x,y,z) = (0,0,0) \end{cases}$$
Writing $(x,y,z) = r(a,b,c)$, with $a^2 + b^2 + c^2 = 1$, we see that the above expression is $r^{2-2\alpha}$ times a bounded quantity, so that $f_x \in C(\mathbb{R}^3)$ whenever $\alpha < 1$. By symmetry, it is also true that $f_y, f_z \in C(\mathbb{R}^3)$.
when $\alpha < 1$, so $f \in C^1(\mathbb{R}^3)$ and consequently $f$ is differentiable on $\mathbb{R}^3$ in this case. In general, the only candidate for $Df(0,0,0)$ is $[0\ 0\ 0]$, and for differentiability at the origin we need to check whether

$$0 = \lim_{(x,y,z)\to(0,0,0)} \frac{f(x,y,z)}{\sqrt{x^2 + y^2 + z^2}} = \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{(x^2 + y^2 + z^2)^{\alpha+1/2}}.$$ 

In polar representation, the above expression becomes $r^{2-2\alpha}abc$ and so the limit does not exist when $\alpha \geq 1$. Thus $f$ is differentiable exactly when $\alpha < 1$. 
