Math 125B, Winter 2015.

**Discussion problems 8**

*Note*. These are problems on differentiable functions.

1. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a function. For each statement below determine, with proof, whether it is true or false.
(a) If $f \in \mathcal{C}^1(\mathbb{R}^2)$, then $f$ is differentiable on $\mathbb{R}^2$.
(b) If $f$ is differentiable on $\mathbb{R}^2$, then $f \in \mathcal{C}^1(\mathbb{R}^2)$.
(c) If $f$ is differentiable on $\mathbb{R}^2$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear, then the function $g : \mathbb{R}^2 \to \mathbb{R}^2$, given by $g(x) = (\sqrt{f(x)^2 + 1}, f(Tx))$ is differentiable.
(d) If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ both exist and are continuous on $\mathbb{R}^2$, and $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$, then the directional derivative $D_u f(0,0) = 0$ for every $u \neq (0,0)$, .

2. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by $f(x_1, x_2, x_3) = (x_1 e^{x_2 + 2x_3}, 2x_2 \cos(x_1 x_3), x_3 e^{4x_1 + 3x_2})$.
(a) Compute $Df(0,0,0)$.
(b) Compute $D(f \circ f)(0,0,0)$. (After solving (a), ten seconds, no computer.)
(c) Is $\frac{\partial^2}{\partial x \partial y} f(0,0) = \frac{\partial^2}{\partial y \partial x} f(0,0)$? (Ten seconds, no computer)

3. Discuss differentiability of $f : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$f(x,y) = \begin{cases} \frac{x^3 - y^3}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(*Remark*. You are encouraged to use the computer for the tedious computing of partial derivatives. If the problem were on an exam, it would include something like the following: "You may assume that $f_x = \frac{x(x^3 + 3xy^2 + 2y^3)}{(x^2 + y^2)^2}$ wherever applicable. Show that $f(x,y) = -f(y,x)$ implies that $f_y(x,y) = -f_x(y,x)$.")

4. Discuss differentiability, in dependence of the parameter $\alpha \in \mathbb{R}$, of $f : \mathbb{R}^3 \to \mathbb{R}$ defined as follows:

$$f(x,y,z) = \begin{cases} \frac{xyz}{(x^2 + y^2 + z^2)^\alpha} & (x,y,z) \neq (0,0,0) \\ 0 & (x,y,z) = (0,0,0) \end{cases}$$

## Brief solutions

1. (a) Yes. This follows from a theorem proved in class.

(b) No. We gave a counterexample in the lecture.

(c) Yes. Both coordinate functions of $g$ are differentiable because of the chain rule. The first coordinate function $g_1 : \mathbb{R}^2 \to \mathbb{R}$ is given by $g_1 = h \circ f$, where $h : \mathbb{R} \to \mathbb{R}$ is the function $h(x) = \sqrt{x^2+1}$ and is differentiable on $\mathbb{R}$. The second coordinate function $g_2 : \mathbb{R}^2 \to \mathbb{R}$ is given by $g_2 = f \circ T$, where $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear and therefore differentiable.

(d) Yes. Under the assumption, $Df(0,0)$ exists and equals $T = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Then $D_u(f)(0,0) = Tu = 0$.

2. (a) By computing partial derivatives

$$Df(0,0,0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) By the chain rule, the answer is the square of the matrix in (a), that is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Yes, this function is in $\mathcal{C}^\infty(\mathbb{R}^2)$, so the two mixed derivatives are equal.

3. Clearly, $f \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$ and so $f$ is differentiable on $\mathbb{R}^2 \setminus \{(0,0)\}$. Also, it is easy to check by polar representation that $f \in \mathcal{C}(\mathbb{R}^2)$. Moreover, $f(x,0) = x$ whenever $x \neq 0$, so

$$f_x(x,y) = \begin{cases} \frac{x(x^3+3xy^2+2y^3)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

As $f_x(0,y) = 0$ for every $y \neq 0$, $f_x$ is clearly not continuous, so we need to check differentiability directly.

We have $f(0,y) = -y$ whenever $y \neq 0$, so $f_y(0,0) = -1$. Thus, the only candidate for $Df(0,0)$ is $\begin{bmatrix} 1 & -1 \end{bmatrix}$, and we need to check whether

$$0 = \lim_{(x,y)\to(0,0)} \frac{f(x,y) - x + y}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{yx^2 - xy^2}{(x^2+y^2)^{3/2}}.$$

If $(x,y) = r(a,b)$, with $a^2 + b^2 = 1$, then the above expression is $ba^2 - ab^2$ and so the limit does not exist. Therefore $f$ is not differentiable at $(0,0)$.

4. It is easy to check by polar representation that $f$ is continuous exactly when $\alpha < 3/2$. Moreover, for all $\alpha$,

$$f_x(x,y,z) = \begin{cases} \frac{yz(x^2 - 2\alpha(x^2+y^2+z^2))}{(x^2+y^2+z^2)^{\alpha+1}} & (x,y,z) \neq (0,0,0) \\ 0 & (x,y,z) = (0,0,0) \end{cases}$$

Writing $(x,y,z) = r(a,b,c)$, with $a^2+b^2+c^2 = 1$, we see that the above expression is $r^{2-2\alpha}$ times a bounded quantity, so that $f_x \in \mathcal{C}(\mathbb{R}^3)$ when $\alpha < 1$. By symmetry, it is also true that $f_y, f_z \in \mathcal{C}(\mathbb{R}^3)$

when $\alpha < 1$, so $f \in \mathcal{C}^1(\mathbb{R}^3)$ and consequently $f$ is differentiable on $\mathbb{R}^3$ in this case. In general, the only candidate for $Df(0,0,0)$ is $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$, and for differentiability at the origin we need to check whether

$$0 = \lim_{(x,y,z)\to(0,0,0)} \frac{f(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{(x^2+y^2+z^2)^{\alpha+1/2}}.$$

In polar representation, the above expression becomes $r^{2-2\alpha}abc$ and so the limit does not exist when $\alpha \geq 1$. Thus $f$ is differentiable exactly when $\alpha < 1$.