Math 125B, Winter 2015.
Mar. 19, 2015.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 8 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 9 pages (including this one) with 8 problems.

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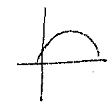
TOTAL	

1. Prove that $\frac{2}{3} \leq \int_0^1 \sqrt{x + 8\sin x}\, dx \leq 2$. (You may use the fact that $\sin x \leq x$ for all $x \geq 0$.)

$\sin x \geqslant 0$ for $x \in [0,1] \subseteq [0,\pi]$



$\sin x \leq x$ for all $x \geqslant 0$

So

$$\sqrt{x} \leq \sqrt{x + 8\sin x} \leq 3\sqrt{x}$$

for $x \in [0,1]$, and so by monotonicity,

$$\int_0^1 \sqrt{x}\,dx \leq \int_0^1 \sqrt{x+8\sin x}\,dx \leq 3\int_0^1 \sqrt{x}\,dx$$

As $\int_0^1 \sqrt{x}\,dx = \frac{2}{3}$,

$$\frac{2}{3} \leq \int_0^1 \sqrt{x+8\sin x}\,dx \leq 2.$$

2. Assume that $f : \mathbb{R} \to \mathbb{R}$ has continuous derivative on $\mathbb{R}$. (*Hint*: integration by parts; in (a), observe the appearance of the derivative of $(f(x))^2$.)

(a) Prove that

$$\int_0^1 x f(x) f'(x)\, dx \le \frac{1}{2} f(1)^2.$$

$$\int_0^1 x f(x) f'(x)\, dx = \int_0^1 x \cdot \frac{d}{dx}\left(\frac{1}{2} f(x)^2\right) dx$$

$$\overset{\text{by parts}}{=} x \cdot \frac{1}{2} f(x)^2 \Big|_0^1 - \underbrace{\int_0^1 \frac{1}{2} f(x)^2 dx}_{\geq 0}$$

$$\leq \frac{1}{2} f(1)^2$$

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(b) Now assume also that $f'(x) \le 3$ for all $x \in [0,1]$. Prove that

$$\int_0^1 x^2 f(x)\, dx \ge \frac{1}{3} f(1) - \frac{1}{4}.$$

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$$\int_0^1 x^2 f(x)\, dx = \int_0^1 f(x) \cdot \frac{d}{dx}\left(\frac{1}{3} x^3\right) dx$$

$$\overset{\text{by parts}}{=} f(x) \cdot \frac{1}{3} x^3 \Big|_0^1 - \int_0^1 f'(x) \cdot \frac{1}{3} x^3 dx$$

$$= \frac{1}{3} f(1) - \frac{1}{3} \int_0^1 \underbrace{f'(x)\, x^3}_{\leq 3x^3} dx$$

$$\overset{\text{by monotonicity}}{\geq} \frac{1}{3} f(1) - \int_0^1 x^3 dx = \frac{1}{3} f(1) - \frac{1}{4}.$$

3. Compute $\lim_{n\to\infty} \int_0^1 \underbrace{\frac{nx^2+\sqrt{n}x^3}{nx^3+x^4+2n}}_{= g_n(x)} dx$.

Let : $g(x) = \dfrac{x^2}{x^3+2}$

$$\sup_{x\in[0,1)} \left| g_n(x) - g(x) \right|$$

$$= \sup_{x\in[0,1)} \frac{\left|\cancel{nx^5} + \sqrt{n}\,x^6 + \cancel{2nx^2} + 2\sqrt{n}\,x^3 - \cancel{nx^5} - x^6 - \cancel{2nx^2}\right|}{(\underbrace{nx^3+x^4}_{\geq 0} + 2n)\,(\underbrace{x^3}_{}+2)_{\geq 2}}$$

$$\leq \frac{\sqrt{n} + 2\sqrt{n} + 1}{2n\cdot 2} = \frac{3\sqrt{n}+1}{4n} \xrightarrow[n\to\infty]{} 0 .$$ 13

Thus $g_n \to g$ uniformly on $[0,1)$, and by the convergence theorem,

$$\int_0^1 g_n(x)\,dx \longrightarrow \int_0^1 g(x)\,dx = \int_0^1 \frac{x^2}{x^3+2}\,dx$$ no way! 12

$$= \int_0^1 \frac{d}{dx}\left(\frac{1}{3}\log(x^3+2)\right) dx \overset{\text{FTC}}{=}$$

$$= \frac{1}{3}\log(x^3+2)\Big|_0^1 = \underline{\underline{\frac{1}{3}\left[\log 3 - \log 2\right]}}$$

4. Determine whether the two improper integrals converge or diverge.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x + \cos x}{x^2}\, dx$

p-int. with p = 2.

$$\left|\frac{\sin^2 x + \cos x}{x^2}\right| \leq \frac{2}{x^2} \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\, dx \text{ conv,}$$

so this integral converges absolutely and thus converges.

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No just.

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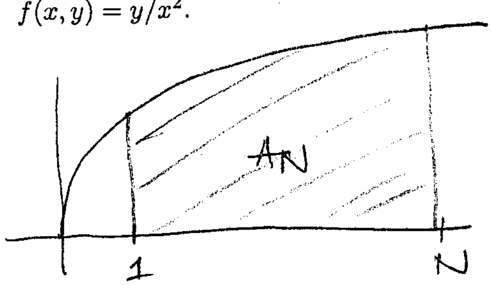
(b) $\displaystyle\int_A f$, where $A \subset \mathbb{R}^2$ is given by $A = \{(x, y) : x > 1, 0 < y < \sqrt{x}\}$ and $f : A \to \mathbb{R}$ is given by $f(x, y) = y/x^2$.



$$\int_A f = \lim_{N\to\infty} \int_{A_N} f$$

convergence of nested subsets

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$A_N = A \cap \{(x,y) : x < N\}$

$f \geq 0$ on A.

$$= \lim_{N\to\infty} \int_1^N dx \int_0^{\sqrt{x}} \frac{y}{x^2}\, dy$$

Fubini (f is cont. on $\overline{A_N}$)

$$= \lim_{N\to\infty} \int_1^N dx \cdot \frac{1}{x^2} \cdot \frac{y^2}{2}\Big|_0^{\sqrt{x}} = \lim_{N\to\infty} \int_1^N \frac{1}{2} \cdot \frac{1}{x}\, dx$$

$$= \frac{1}{2}\int_1^\infty \frac{1}{x}\, dx = \infty$$

diverges

p-integral with p = 1

5. For each statement below determine, with proof, whether it is true or false.

(a) If $f : \mathbb{R}^2 \to \mathbb{R}$ is in $C^1(\mathbb{R}^2)$, then f is differentiable on $\mathbb{R}^2$.

Yes. This is a theorem from class.

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(b) If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous on $\mathbb{R}^2$, and

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0,$$

then f is differentiable at $(0,0)$, with $Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

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$$f(0,0) \underset{\text{continuity}}{=} \lim_{(x,y)\to(0,0)} f(x,y)$$

$$= \lim_{(x,y)\to(0,0)} \underbrace{\frac{f(x,y)}{\sqrt{x^2+y^2}}}_{\to 0} \cdot \underbrace{\sqrt{x^2+y^2}}_{\to 0} = 0,$$

Thus, we need to check whether

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - Df(0,0)\begin{bmatrix} x \\ y \end{bmatrix}}{\sqrt{x^2+y^2}} = 0$$

but this is exactly the above limit. Yes.

$f \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$.

6. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is given by:

$$f(x,y) = \begin{cases} \frac{x^2y-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Is f continuous on $\mathbb{R}^2$? Write $(x,y) = r(a,b)$, $a^2+b^2=1$, so $|a|,|b| \leq$

$$\lim_{(x,y)\to(0,0)} \left| \frac{x^2y-y^3}{x^2+y^2} \right| = \lim_{(x,y)\to(0,0)} \left| \frac{r^3(a^2b-b^3)}{r^2} \right|$$

$$= \lim_{(x,y)\to(0,0)} r \cdot |a^2b-b^3| \leq \lim_{(x,y)\to(0,0)} 2r = 0$$

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Yes.

(b) Is f differentiable on $\mathbb{R}^2$? (*Hint*. You do not need to compute partial derivatives away from $(0,0)$ to answer this question.)

$x \neq 0$: $f(0,y) = -y$, so $\frac{\partial}{\partial y} f(0,0) = -1$

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$y \neq 0$: $f(x,0) = 0$, so $\frac{\partial}{\partial x} f(0,0) = 0$.

Need to check whether this goes to 0 as $(x,y)\to(0,0)$:

$$\frac{f(x,y) - f(0,0) - [0 \;\; -1] \begin{bmatrix} x \\ y \end{bmatrix}}{\sqrt{x^2+y^2}}$$

$$= \frac{f(x,y)+y}{\sqrt{x^2+y^2}} = \frac{x^2y - y^3 + x^2y + y^3}{(x^2+y^2)^{3/2}} = \frac{\cancel{r^3} 2a^2b}{\cancel{r^3}} = 2a^2b,$$

so $\lim_{(x,y)\to(0,0)}$ does not exist, NO.

7. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (x^2 - 2y, xy + 1)$.
(a) Describe the set of points (x,y) at which f has a differentiable local inverse.

$$Df = \begin{bmatrix} 2x & -2 \\ y & x \end{bmatrix}$$

$$\det Df = 2x^2 + 2y = 2(y + x^2)$$

f has differentiable local inverse everywhere but on the parabola $y = -x^2$.

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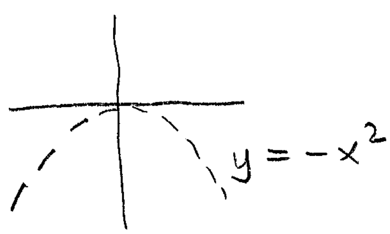

(b) Show, using the computation from (a), that f has a local inverse at $(1,0)$; call this local inverse g. Assume that $h : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $h(x,y) = (2x+3, x+2y)$. Compute $D(h \circ g)(f(1,0))$.

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At $(1,0)$ $\det Df = 2 \neq 0$

$Dh = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ at all points of $\mathbb{R}^2$

$$D(h \circ g)(f(1,0)) \underset{\substack{\uparrow \\ \text{Chain rule}}}{=} Dh \cdot Dg(f(1,0))$$

$$\underset{\substack{\uparrow \\ \because \text{ Inverse function Thm.}}}{=} Dh \cdot Df(1,0)^{-1} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 1 & 2 \\ 1/2 & 3 \end{bmatrix}}}$$

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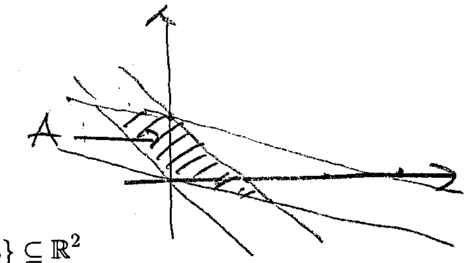

8. Let

$$A = \{(x, y) : 0 \leq x + y \leq 1, 0 \leq x + 3y \leq 3\} \subseteq \mathbb{R}^2$$

and let $f : A \to \mathbb{R}$ be given by $f(x, y) = \sqrt{x + y}$. Compute $\int_A f$. (*Hint*: Change of variables.)

$u = x+y$

$v = x+3y$

$\begin{bmatrix} u \\ v \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix} \quad T = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

$g = T^{-1}$, $Dg = T^{-1}$

(1) $\det Dg = \det T^{-1} = \dfrac{1}{\det T} = \dfrac{1}{2}$

(2) Region in (u,v): $\{(u,v) : 0 \leq u \leq 1, 0 \leq v \leq 3\}$

(3) Function in (u,v): $\sqrt{u}$

By Change of variables Thm, and Fubini:

$$\int_A f = \int_0^1 du \int_0^3 \sqrt{u} \cdot \frac{1}{2}\, dv$$

$$= \frac{3}{2} \cdot \int_0^1 \sqrt{u}\, du = \frac{3}{2} \cdot \frac{2}{3} = \underline{\underline{1}}.$$