to extend the integral to the case $a > b$. In particular, if $f(x)$ is integrable and nonpositive on $[a, b]$, then the area of the region bounded by the curves $y = f(x)$, $y = 0$, $x = a$, and $x = b$ is given by $\int_b^a f(x)dx$.

In the next section we shall use the machinery of upper and lower sums to prove several familiar theorems about the Riemann integral. We close this section with one more result which reinforces the connection between integration and area.

5.16 Theorem. *If $f(x) = \alpha$ is constant on $[a, b]$, then*

$$\int_a^b f(x)\,dx = \alpha(b - a).$$

Proof. By Theorem 5.10, f is integrable on $[a, b]$. Hence, it follows from Theorem 5.15 and Remark 5.5 that

$$\int_a^b f(x)\,dx = (U)\int_a^b f(x)\,dx = \inf_P U(f, P) = \alpha(b - a). \qquad \blacksquare$$

EXERCISES

5.1.0. Suppose that $a < b < c$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f is Riemann integrable on $[a, b]$, then f is continuous on $[a, b]$.
b) If $|f|$ is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$.
c) For all bounded functions $f : [a, b] \to \mathbf{R}$,

$$(L)\int_a^b f(x)\,dx \le \int_a^b f(x)\,dx \le (U)\int_a^b f(x)\,dx.$$

d) If f is continuous on $[a, b)$ and on $[b, c]$, then f is Riemann integrable on $[a, c]$.

5.1.1. For each of the following, compute $U(f, P)$, $L(f, P)$, and $\int_0^2 f(x)dx$, where

$$P = \left\{0, \frac{1}{2}, 1, 2\right\}.$$

Find out whether the lower sum or the upper sum is a better approximation to the integral. Graph f and explain why this is so.

a) $f(x) = x^3$
b) $f(x) = 3 - x^2$
c) $f(x) = \sin(x/5)$

5.1.2. a) Prove that for each $n \in \mathbf{N}$,

$$P_n := \left\{ \frac{j}{n} : j = 0, 1, \ldots, n \right\}$$

is a partition of $[0, 1]$.

b) Prove that a bounded function f is integrable on $[0, 1]$ if

$$(*) \qquad I_0 := \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n),$$

in which case $\int_0^1 f(x)dx$ equals I_0.

c) For each of the following functions, use Exercise 1.4.4 to find formulas for the upper and lower sums of f on P_n, and use them to compute the value of $\int_0^1 f(x)dx$.

α)
$$f(x) = x$$

β)
$$f(x) = x^2$$

γ)
$$f(x) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$$

5.1.3. Let $E = \{1/n : n \in \mathbf{N}\}$. Prove that the function

$$f(x) = \begin{cases} 1 & x \in E \\ 0 & \text{otherwise} \end{cases}$$

is integrable on $[0, 1]$. What is the value of $\int_0^1 f(x)dx$?

$\boxed{\textbf{5.1.4}}$. **This exercise is used in Section *14.2.** Suppose that $a < b$ and that $f : [a, b] \to \mathbf{R}$ is bounded.

a) Prove that if f is continuous at $x_0 \in [a, b]$ and $f(x_0) \neq 0$, then

$$(L) \int_a^b |f(x)|\, dx > 0.$$

b) Show that if f is continuous on $[a, b]$, then $\int_a^b |f(x)|dx = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

c) Does part b) hold if the absolute values are removed? If it does, prove it. If it does not, provide a counterexample.

5.1.5. Suppose that $a < b$ and that $f : [a, b] \to \mathbf{R}$ is continuous. Show that

$$\int_a^c f(x)\, dx = 0$$

for all $c \in [a, b]$ if and only if $f(x) = 0$ for all $x \in [a, b]$. (Compare with Exercise 5.1.4, and notice that f need not be nonnegative here.)

5.1.6. Let f be integrable on $[a, b]$ and E be a finite subset of $[a, b]$. Show that if g is a bounded function which satisfies $g(x) = f(x)$ for all $x \in [a, b] \backslash E$, then g is integrable on $[a, b]$ and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

$\boxed{5.1.7}$. **This exercise is used in Section 12.3.** Let f, g be bounded on $[a, b]$.

a) Prove that

$$(U)\int_a^b (f(x) + g(x))\,dx \le (U)\int_a^b f(x)\,dx + (U)\int_a^b g(x)\,dx$$

and

$$(L)\int_a^b (f(x) + g(x))\,dx \ge (L)\int_a^b f(x)\,dx + (L)\int_a^b g(x)\,dx.$$

b) Prove that

$$(U)\int_a^b f(x)\,dx = (U)\int_a^c f(x)\,dx + (U)\int_c^b f(x)\,dx$$

and

$$(L)\int_a^b f(x)\,dx = (L)\int_a^c f(x)\,dx + (L)\int_c^b f(x)\,dx$$

for $a < c < b$.

$\boxed{5.1.8}$. **This exercise is used in Sections *5.5, 6.2, and *7.5.**

a) If f is increasing on $[a, b]$ and $P = \{x_0, \dots, x_n\}$ is any partition of $[a, b]$, prove that

$$\sum_{j=1}^{n}(M_j(f) - m_j(f))\,\Delta x_j \le (f(b) - f(a))\,\|P\|.$$

b) Prove that if f is monotone on $[a, b]$, then f is integrable on $[a, b]$. [**Note**: By Theorem 4.19, f has at most countably many (i.e., relatively few) discontinuities on $[a, b]$. This has nothing to do with the proof of part b), but points out a general principle which will be discussed in Section 9.6.]

5.1.9. Let $a < b$ and $0 < c < d$ be real numbers and $f : [a, b] \to [c, d]$. If f is Riemann integrable on $[a, b]$, prove that $\sqrt{f}$ is Riemann integrable on $[a, b]$.

5.1.10. Let f be bounded on a nondegenerate interval $[a, b]$. Prove that f is integrable on $[a, b]$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that

$$P \supseteq P_\varepsilon \quad \text{implies} \quad |U(f, P) - L(f, P)| < \varepsilon.$$

5.2 RIEMANN SUMS

There is another definition of the Riemann integral frequently found in elementary calculus texts.

5.17 Definition.

Let $f : [a, b] \to \mathbf{R}$.

i) A *Riemann sum* of f with respect to a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

$$\mathcal{S}(f, P, t_j) := \sum_{j=1}^{n} f(t_j)\, \Delta x_j.$$

ii) The Riemann sums of f are said to *converge* to $I(f)$ as $\|P\| \to 0$ if and only if given $\varepsilon > 0$ there is a partition P_ε of $[a, b]$ such that

$$P = \{x_0, \dots, x_n\} \supseteq P_\varepsilon \quad \text{implies} \quad \left|\mathcal{S}(f, P, t_j) - I(f)\right| < \varepsilon$$

for all choices of $t_j \in [x_{j-1}, x_j]$, $j = 1, 2, \dots, n$. In this case we shall use the notation

$$I(f) = \lim_{\|P\| \to 0} \mathcal{S}(f, P, t_j) := \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\, \Delta x_j.$$

The following result shows that this definition of the Riemann integral is the same as the one using upper and lower integrals.

5.18 Theorem. *Let $a, b \in \mathbf{R}$ with $a < b$, and suppose that $f : [a, b] \to \mathbf{R}$. Then f is Riemann integrable on $[a, b]$ if and only if*

$$I(f) = \lim_{\|P\| \to 0} \sum_{j=1}^{n} f(t_j)\, \Delta x_j$$

exists, in which case $I(f) = \int_a^b f(x)dx$.