Math 125B, Winter 2015.

Homework 1 Solutions

5.1.0. (a) No. For example $f:[0,1]\to\mathbb{R}$, given by $f(x)=\begin{cases}0 & x\in[0,1)\\ 1 & x=1\end{cases}$, is integrable (with integral 0, as we showed in the lecture) but is not continuous.
(b) No. For example, $f:[0,1]\to\mathbb{R}$, given by $f(x)=\begin{cases}-1 & x\in\mathbb{Q}\\ 1 & x\notin\mathbb{Q}\end{cases}$, has $U(f)=1$ and $L(f)=-1$, as is easily shown in the same fashion as for the Dirichlet function. Thus f is not Riemann integrable, but $|f|\equiv 1$ is a constant function and consequently is Riemann integrable.
(c) No. This is somewhat of a trick question: the integral $\int_a^b f(x)\,dx$ does not necessarily exist under the assumptions.
(d) No. The assumptions do not imply that f is bounded on $[a,b]$; for example $f:[-1,1]\to\mathbb{R}$, given by $f(x)=\begin{cases}-1/x & x\in[-1,0)\\ 0 & x\in[0,1]\end{cases}$, is not bounded above.

5.1.7. (a) In this part, I will use $U(f)$ instead of $(U)\int_a^b f(x)\,dx$, and analogous notation for other functions.

For any set A and real-valued functions f and g on A, we have, for any $x\in A$,

$$f(x)+g(x)\le\sup_A f+\sup_A g$$

and so

$$\sup_A(f+g)\le\sup_A f+\sup_A g.$$

It follows that for any partition P given by $a=x_0<x_1<\cdots<x_n=b$,

$$\begin{aligned}U(f+g,P)&=\sum_{j=1}^n\sup_{[x_{j-1},x_j]}(f+g)\,\Delta x_j\\&\le\sum_{j=1}^n\left(\sup_{[x_{j-1},x_j]}f+\sup_{[x_{j-1},x_j]}g\right)\Delta x_j\\&=\sum_{j=1}^n\sup_{[x_{j-1},x_j]}f\,\Delta x_j+\sum_{j=1}^n\sup_{[x_{j-1},x_j]}g\,\Delta x_j\\&=U(f,P)+U(g,P).\end{aligned}$$

Take any partitions P_1 and P_2. We need to show that

$$U(f+g)\le U(f,P_1)+U(g,P_2).$$

Then it will follow that $U(f+g)\le\inf_{P_1}U(f,P_1)+\inf_{P_2}U(g,P_2)=U(f)+U(g)$. To show the above inequality, take $P=P_1\cup P_2$; then

$$U(f+g)\le U(f+g,P)\le U(f,P)+U(g,P)\le U(f,P_1)+U(g,P_2).$$

The proof for the lower integral is very similar (one can also use that $L(f) = -U(-f)$).
(b) Again, we prove this only for the upper integral. First we prove the ($\geq$) part. Take any partition P of $[a, b]$. Take the refinement $P' = P \cup \{c\}$. Then $P_1 = P' \cap [a, c]$ is a partition of $[a, c]$ and $P_2 = P' \cap [c, b]$ is a partition of $[c, b]$. Also, we can divide the sum for $U(f, P')$ into terms up to c and terms from c on to get $U(f, P') = U(f, P_1) + U(f, P_2)$. Then

$$U(f, P) \geq U(f, P') = U(f, P_1) + U(f, P_2) \geq (U) \int_a^c f + (U) \int_c^b f$$

and, as this is true for arbitrary P,

$$(U) \int_a^b f \geq (U) \int_a^c f + (U) \int_c^b f.$$

Next we prove the ($\leq$) part. Take any partitions P_1 of $[a, c]$ and P_2 of $[c, b]$. Then $P = P_1 \cup P_2$ is a partition of $[a, b]$. Then,

$$(U) \int_a^b f \leq U(f, P) = U(f, P_1) + U(f, P_2),$$

and then after taking the infimum over P_1 and P_2,

$$(U) \int_a^b f \leq (U) \int_a^c f + (U) \int_c^b f.$$