EXERCISES

5.2.0. Suppose that $a < b$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f and g are Riemann integrable on $[a, b]$, then $f - g$ is Riemann integrable on $[a, b]$.

b) If f is Riemann integrable on $[a, b]$ and P is any polynomial on **R**, then $P \circ f$ is Riemann integrable on $[a, b]$.

c) If f and g are nonnegative real functions on $[a, b]$, with f continuous and g Riemann integrable on $[a, b]$, then there exist $x_0, x_1 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_{x_1}^b g(x)\,dx.$$

d) If f and g are Riemann integrable on $[a, b]$ and f is continuous, then there is an $x_0 \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(x_0)\int_a^b g(x)\,dx.$$

5.2.1. Using the connection between integrals and area, evaluate each of the following integrals.

a) $$\int_{-2}^{2} |x+1|\,dx$$

b) $$\int_{-2}^{2} (|x+1| + |x|)\,dx$$

c) $$\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx, \qquad a > 0$$

d) $$\int_{0}^{2} (5 + \sqrt{2x + x^2})\,dx$$

5.2.2. a) Suppose that $a < b$ and $n \in \mathbf{N}$ is even. If f is continuous on $[a, b]$ and $\int_a^b f(x)x^n dx = 0$, prove that $f(x) = 0$ for at least one $x \in [a, b]$.

b) Show that part a) might not be true if n is odd.

c) Prove that part a) does hold for odd n when $a \geq 0$.

5.2.3. Use Taylor polynomials with three or four nonzero terms to verify the following inequalities.

a) $$0.3095 < \int_0^1 \sin(x^2)\,dx < 0.3103$$

(The value of this integral is approximately 0.3102683.)

b)
$$1.4571 < \int_0^1 e^{x^2}\,dx < 1.5704$$

(The value of this integral is approximately 1.4626517.)

5.2.4. Suppose that $f : [0, \infty) \to [0, \infty)$ is integrable on every closed interval $[a, b] \subset [0, \infty)$. If

$$F(x) := \int_0^x e^{-y^2} f(y)\,dy, \quad x \in [0, \infty),$$

then there is a function $g : [0, \infty) \to [0, \infty)$ such that $F(x) = \int_{g(x)}^x f(y)\,dy$ for all $x \in [0, \infty)$.

5.2.5. Prove that if f is integrable on $[0, 1]$ and $\beta > 0$, then

$$\lim_{n\to\infty} n^\alpha \int_0^{1/n^\beta} f(x)\,dx = 0$$

for all $\alpha < \beta$.

5.2.6. a) Suppose that $g_n \geq 0$ is a sequence of integrable functions which satisfies

$$\lim_{n\to\infty} \int_a^b g_n(x)\,dx = 0.$$

Show that if $f : [a, b] \to \mathbf{R}$ is integrable on $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b f(x)g_n(x)\,dx = 0.$$

b) Prove that if f is integrable on $[0, 1]$, then

$$\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0.$$

5.2.7. Suppose that f is integrable on $[a, b]$, that $x_0 = a$, and that x_n is a sequence of numbers in $[a, b]$ such that $x_n \uparrow b$ as $n \to \infty$. Prove that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=0}^{n} \int_{x_k}^{x_{k+1}} f(x)\,dx.$$

5.2.8. Let f be continuous on a closed, nondegenerate interval $[a, b]$ and set

$$M = \sup_{x\in[a,b]} |f(x)|.$$

a) Prove that if $M > 0$ and $p > 0$, then for every $\varepsilon > 0$ there is a nondegenerate interval $I \subset [a, b]$ such that

$$(M-\varepsilon)^p|I| \le \int_a^b |f(x)|^p\,dx \le M^p(b-a).$$

b) Prove that

$$\lim_{p\to\infty}\left(\int_a^b |f(x)|^p\,dx\right)^{1/p} = M.$$

5.2.9. Let $f : [a, b] \to \mathbf{R}$, $a = x_0 < x_1 < \cdots < x_n = b$, and suppose that $f(x_k+)$ exists and is finite for $k = 0, 1, \ldots, n-1$ and $f(x_k-)$ exists and is finite for $k = 1, \ldots, n$. Show that if f is continuous on each subinterval (x_{k-1}, x_k), then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \sum_{k=1}^n \int_{x_{k-1}}^{x_k} f(x)\,dx.$$

5.2.10. Prove that if f and g are integrable on $[a, b]$, then so are $f \vee g$ and $f \wedge g$ (see Exercise 3.1.8).

5.2.11. Suppose that $f : [a, b] \to \mathbf{R}$.

a) If f is not bounded above on $[a, b]$, then given any partition P of $[a, b]$ and $M > 0$, there exist $t_j \in [x_{j-1}, x_j]$ such that $\mathcal{S}(f, P, t_j) > M$.

b) If the Riemann sums of f converge to a finite number $I(f)$, as $\|P\| \to 0$, then f is bounded on $[a, b]$.

5.3 THE FUNDAMENTAL THEOREM OF CALCULUS

Let f be integrable on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$. By Theorem 5.26, F is continuous on $[a, b]$. The next result shows that if f is continuous, then F is continuously differentiable. Thus "indefinite integration" improves the behavior of the function.

5.28 Theorem. [FUNDAMENTAL THEOREM OF CALCULUS].
Let $[a, b]$ be nondegenerate and suppose that $f : [a, b] \to \mathbf{R}$.

i) *If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, then $F \in \mathcal{C}^1[a, b]$ and*

$$\frac{d}{dx}\int_a^x f(t)\,dt := F'(x) = f(x)$$

for each $x \in [a, b]$.

ii) *If f is differentiable on $[a, b]$ and f' is integrable on $[a, b]$, then*

$$\int_a^x f'(t)\,dt = f(x) - f(a)$$

for each $x \in [a, b]$.