EXERCISES

5.2.0. Suppose that \( a < b \). Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If \( f \) and \( g \) are Riemann integrable on \([a, b]\), then \( f - g \) is Riemann integrable on \([a, b]\).

b) If \( f \) is Riemann integrable on \([a, b]\) and \( P \) is any polynomial on \( \mathbb{R} \), then \( P \circ f \) is Riemann integrable on \([a, b]\).

c) If \( f \) and \( g \) are nonnegative real functions on \([a, b]\), with \( f \) continuous and \( g \) Riemann integrable on \([a, b]\), then there exist \( x_0, x_1 \in [a, b] \) such that

\[
\int_a^b f(x)g(x) \, dx = f(x_0) \int_{x_1}^b g(x) \, dx.
\]

d) If \( f \) and \( g \) are Riemann integrable on \([a, b]\) and \( f \) is continuous, then there is an \( x_0 \in [a, b] \) such that

\[
\int_a^b f(x)g(x) \, dx = f(x_0) \int_a^b g(x) \, dx.
\]

5.2.1. Using the connection between integrals and area, evaluate each of the following integrals.

a) \[
\int_{-2}^2 |x + 1| \, dx
\]

b) \[
\int_{-2}^2 (|x + 1| + |x|) \, dx
\]

c) \[
\int_{-a}^a \sqrt{a^2 - x^2} \, dx, \quad a > 0
\]

d) \[
\int_0^a (5 + \sqrt{2x + x^2}) \, dx
\]

5.2.2. a) Suppose that \( a < b \) and \( n \in \mathbb{N} \) is even. If \( f \) is continuous on \([a, b]\) and \( \int_a^b f(x)x^n \, dx = 0 \), prove that \( f(x) = 0 \) for at least one \( x \in [a, b] \).

b) Show that part a) might not be true if \( n \) is odd.

c) Prove that part a) does hold for odd \( n \) when \( a \geq 0 \).

5.2.3. Use Taylor polynomials with three or four nonzero terms to verify the following inequalities.

a) \[
0.3095 < \int_0^1 \sin(x^2) \, dx < 0.3103
\]

(The value of this integral is approximately 0.3102683.)
b) \[ 1.4571 < \int_0^1 e^{x^2} \, dx < 1.5704 \]

(The value of this integral is approximately 1.4626517.)

5.2.4. Suppose that \( f : [0, \infty) \to [0, \infty) \) is integrable on every closed interval \([a, b] \subset [0, \infty)\). If

\[ F(x) := \int_0^x e^{-y^2} f(y) \, dy, \quad x \in [0, \infty), \]

then there is a function \( g : [0, \infty) \to [0, \infty) \) such that \( F(x) = \int_{g(x)}^x f(y) \, dy \) for all \( x \in [0, \infty) \).

5.2.5. Prove that if \( f \) is integrable on \([0, 1]\) and \( \beta > 0 \), then

\[ \lim_{n \to \infty} n^\alpha \int_0^{1/n^\beta} f(x) \, dx = 0 \]

for all \( \alpha < \beta \).

5.2.6. a) Suppose that \( g_n \geq 0 \) is a sequence of integrable functions which satisfies

\[ \lim_{n \to \infty} \int_a^b g_n(x) \, dx = 0. \]

Show that if \( f : [a, b] \to \mathbb{R} \) is integrable on \([a, b]\), then

\[ \lim_{n \to \infty} \int_a^b f(x) g_n(x) \, dx = 0. \]

b) Prove that if \( f \) is integrable on \([0, 1]\), then

\[ \lim_{n \to \infty} \int_0^1 x^n f(x) \, dx = 0. \]

5.2.7. Suppose that \( f \) is integrable on \([a, b]\), that \( x_0 = a \), and that \( x_n \) is a sequence of numbers in \([a, b]\) such that \( x_n \uparrow b \) as \( n \to \infty \). Prove that

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{k=0}^n \int_{x_k}^{x_{k+1}} f(x) \, dx. \]

5.2.8. Let \( f \) be continuous on a closed, nondegenerate interval \([a, b]\) and set

\[ M = \sup_{x \in [a, b]} |f(x)|. \]
a) Prove that if $M > 0$ and $p > 0$, then for every $\varepsilon > 0$ there is a nondegenerate interval $I \subset [a, b]$ such that

\[ (M - \varepsilon)^p |I| \leq \int_a^b |f(x)|^p \, dx \leq M^p (b - a). \]

b) Prove that

\[ \lim_{p \to \infty} \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} = M. \]

5.2.9. Let $f : [a, b] \to \mathbb{R}$, $a = x_0 < x_1 < \cdots < x_n = b$, and suppose that $f(x_k^+) \text{ exists and is finite for } k = 0, 1, \ldots, n - 1$ and $f(x_k^-) \text{ exists and is finite for } k = 1, \ldots, n$. Show that if $f$ is continuous on each subinterval $(x_{k-1}, x_k)$, then $f$ is integrable on $[a, b]$ and

\[ \int_a^b f(x) \, dx = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f(x) \, dx. \]

5.2.10. Prove that if $f$ and $g$ are integrable on $[a, b]$, then so are $f \vee g$ and $f \wedge g$ (see Exercise 3.1.8).

5.2.11. Suppose that $f : [a, b] \to \mathbb{R}$.

a) If $f$ is not bounded above on $[a, b]$, then given any partition $P$ of $[a, b]$ and $M > 0$, there exist $t_j \in [x_{j-1}, x_j]$ such that $S(f, P, t_j) > M$.

b) If the Riemann sums of $f$ converge to a finite number $I(f)$, as $\|P\| \to 0$, then $f$ is bounded on $[a, b]$.

5.3 THE FUNDAMENTAL THEOREM OF CALCULUS

Let $f$ be integrable on $[a, b]$ and $F(x) = \int_a^x f(t) \, dt$. By Theorem 5.26, $F$ is continuous on $[a, b]$. The next result shows that if $f$ is continuous, then $F$ is continuously differentiable. Thus "indefinite integration" improves the behavior of the function.

5.28 Theorem. [FUNDAMENTAL THEOREM OF CALCULUS].

Let $[a, b]$ be nondegenerate and suppose that $f : [a, b] \to \mathbb{R}$.

i) If $f$ is continuous on $[a, b]$ and $F(x) = \int_a^x f(t) \, dt$, then $F \in C^1[a, b]$ and

\[ \frac{d}{dx} \int_a^x f(t) \, dt := F'(x) = f(x) \]

for each $x \in [a, b]$.

ii) If $f$ is differentiable on $[a, b]$ and $f'$ is integrable on $[a, b]$, then

\[ \int_a^x f'(t) \, dt = f(x) - f(a) \]

for each $x \in [a, b]$. 