Math 125B, Winter 2015.

Homework 2 Solutions

5.1.9. As $g(x) = \sqrt{x}$ is a continuous function on $[c, d]$ (which is true as soon as $c \geq 0$, so you do not need the strict inequality $c > 0$), and f is Riemann integrable, the composite function $g \circ f = \sqrt{f}$ is Riemann integrable.

5.2.0. (a) Yes, by linearity.
(b) Yes. A polynomial is continuous function on every interval, so this is true by composition with a continuous function.
(d) No. We will find an example with $\int_{-1}^{1} g = 0$ while $\int_{-1}^{1} fg \neq 0$. For example, take $g(x) = \begin{cases} -1 & x \in [-1, 0) \\ 1 & x \in [0, 1] \end{cases}$ and $f(x) = x$. Then $f(x)g(x) = |x|$. Clearly $\int_{-1}^{1} g = 0$ (easily computed as g is piecewise constant), and $\int_{-1}^{1} fg = 1$ (by dividing into two pieces, each with integral 1/2).

5.2.8. (a) By replacing f by $|f|$, we may assume $f \geq 0$. By continuity, there exists an $x_0 \in [a, b]$ such that $f(x_0) = M$. Also by continuity, there exists a $\delta > 0$ so that $M - f(x) = f(x_0) - f(x) < \epsilon$ when $x \in [a, b]$ and $|x - x_0| < \delta$. Take $I = [x_0 - \delta/2, x_0 + \delta/2] \cap [a, b]$ (which has positive length). Then $f \geq M - \epsilon$ on I, so that $f^p \geq (M - \epsilon)^p$ on I. Further, as $f \geq 0$, $\int_a^b f^p \geq \int_I f^p$. Therefore, by the Comparison Theorem,

$$\underline{\int_a^b} f^p \geq \int_a^b f^p \geq \int_I f^p \geq (M - \epsilon)^p |I|.$$

On the other hand, on $[a, b]$, $f \leq M$ and $f^p \leq M^p$, and so by the Comparison Theorem,

$$\int_a^b f^p \leq M^p (b - a).$$

(b) We have, for any $\epsilon > 0$,

$$\liminf_p \left(\int_a^b f^p \right)^{1/p} \geq (M - \epsilon) \lim_p |I|^{1/p} = M - \epsilon$$

so that

$$\liminf_p \left(\int_a^b f^p \right)^{1/p} \geq M.$$

Similarly

$$\limsup_p \left(\int_a^b f^p \right)^{1/p} \leq M \lim_p (b - a)^{1/p} = M.$$

Therefore, the limit exists and equals M.

5.2.10. The equalities $\max(f, g) = (|f - g| + f + g)/2$ and $\min(f, g) = f + g - \max(f, g)$ are easily proved by checking that they are true when $f(x) \leq g(x)$ and when $g(x) \leq f(x)$. Then the claimed integrability follows from the fact that sums, differences and absolute values of integrable functions are integrable.