EXERCISES

5.3.0. Suppose that $a < b$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f is continuous and nonnegative on $[a, b]$ and $g : [a, b] \to [a, b]$ is differentiable and increasing on $[a, b]$, then

$$F(x) := \int_a^{g(x)} f(t)\, dt$$

is increasing on $[a, b]$.

b) If f and g are differentiable on $[a, b]$, if f' and g' are Riemann integrable on $[a, b]$, and if $f(a) = 0$ but g is never zero on $[a, b]$, then

$$f(x) = \int_a^x g(t) \left(\frac{f(t)}{g(t)} \right)' dt + \int_a^x \frac{f(t)g'(t)}{g(t)} dt$$

for all $x \in [a, b]$.

c) If f and g are differentiable on $[a, b]$, and if f' and g' are Riemann integrable on $[a, b]$, then

$$\int_a^b f'(x)g(x)\, dx + \int_a^b f(x)g'(x)\, dx = 0$$

if and only if $f(a)g(a) = f(b)g(b)$.

d) If f and g are continuously differentiable on $[a, b]$, and if h is continuous on $[a, b]$, then

$$\int_{g(f(a))}^{g(f(b))} h(x)\, dx = \int_a^b h(g(f(x)))g'(f(x))f'(x)\, dx.$$

5.3.1. If $f : \mathbf{R} \to \mathbf{R}$ is continuous, find $F'(x)$ for each of the following functions.

a) $$F(x) = \int_{x^2}^1 f(t)\, dt$$

b) $$F(x) = \int_{x^2}^{x^3} f(t)\, dt$$

c) $$F(x) = \int_0^{x \cos x} t f(t)\, dt$$

d) $$F(x) = \int_0^x f(t - x)\, dt$$

5.3.2. Suppose that f is nonnegative and continuous on $[1, 2]$ and that $\int_1^2 x^k f(x)dx = 5 + k^2$ for $k = 0, 1, 2$. Prove that each of the following statements is correct.

a) $$\int_1^4 f(\sqrt{x})\, dx = 12$$

b) $$\int_{\sqrt{2}/2}^1 f\left(\frac{1}{x^2}\right) dx \le \frac{5}{2}$$

c) $$\int_0^1 x^2 f(x+1)\, dx = 2$$

5.3.3. Suppose that f is integrable on $[0.5, 2]$ and that

$$\int_{0.5}^1 x^k f(x)\, dx = \int_1^2 x^k f(x)\, dx + 2k^2 = 3 + k^2$$

for $k = 0, 1, 2$. Compute the exact values of each of the following integrals.

a) $$\int_0^1 x^3 f(x^2+1)\, dx$$

b) $$\int_0^{\sqrt{3}/2} \frac{x^3}{\sqrt{1-x^2}} f\left(\sqrt{1-x^2}\right) dx$$

5.3.4. Suppose that f and g are differentiable on $[0, e]$ and that f' and g' are integrable on $[0, e]$.

a) If $\int_1^e f(x)/x\, dx < f(e)$, prove that

$$\int_1^e f'(x) \log x\, dx > 0.$$

b) If $f(0) = f(1) = 0$, prove that

$$\int_0^1 e^x (f(x) + f'(x))\, dx = 0.$$

c) If $0 \in \{f(0), g(0)\} \cap \{f(e), g(e)\}$, prove that

$$\int_0^e f(x)g'(x)\, dx = -\int_0^e g(x)f'(x)\, dx.$$

5.3.5. Use the First Mean Value Theorem for Integrals to prove the following version of the Mean Value Theorem for Derivatives. If $f \in \mathcal{C}^1[a, b]$, then there is an $x_0 \in [a, b]$ such that

$$f(b) - f(a) = (b - a) f'(x_0).$$

5.3.6. If f is continuous on $[a, b]$ and there exist numbers $\alpha \neq \beta$ such that

$$\alpha \int_a^c f(x)\, dx + \beta \int_c^b f(x)\, dx = 0$$

holds for all $c \in (a, b)$, prove that $f(x) = 0$ for all $x \in [a, b]$.

5.3.7. **This exercise is used in Sections 5.4 and 6.1.** Define $L : (0, \infty) \to \mathbf{R}$ by

$$L(x) = \int_1^x \frac{dt}{t}.$$

a) Prove that L is differentiable and strictly increasing on $(0, \infty)$, with $L'(x) = 1/x$ and $L(1) = 0$.
b) Prove that $L(x) \to \infty$ as $x \to \infty$ and $L(x) \to -\infty$ as $x \to 0+$. (You may wish to prove

$$L(2^n) = \sum_{k=1}^{n} \int_{2^{k-1}}^{2^k} \frac{dt}{t} > \sum_{k=1}^{n} 2^{-k}\left(2^k - 2^{k-1}\right) = \frac{n}{2}$$

for all $n \in \mathbf{N}$.)
c) Using the fact that $(x^q)' = qx^{q-1}$ for $x > 0$ and $q \in \mathbf{Q}$ (see Exercise 4.2.7), prove that $L(x^q) = qL(x)$ for all $q \in \mathbf{Q}$ and $x > 0$.
d) Prove that $L(xy) = L(x) + L(y)$ for all $x, y \in (0, \infty)$.
e) Suppose that $e = \lim_{n\to\infty}(1 + 1/n)^n$ exists. (It does—see Example 4.22.) Use l'Hôpital's Rule to show that $L(e) = 1$. [$L(x)$ is the *natural logarithm* function $\log x$.]

5.3.8. **This exercise was used in Section 4.3.** Let $E = L^{-1}$ represent the inverse function of L, where L is defined in Exercise 5.3.7.

a) Use the Inverse Function Theorem to show that E is differentiable and strictly increasing on $\mathbf{R}$ with $E'(x) = E(x)$, $E(0) = 1$, and $E(1) = e$.
b) Prove that $E(x) \to \infty$ as $x \to \infty$ and $E(x) \to 0$ as $x \to -\infty$.
c) Prove that $E(xq) = (E(x))^q$ and $E(q) = e^q$ for all $q \in \mathbf{Q}$ and $x \in \mathbf{R}$.
d) Prove that $E(x + y) = E(x)E(y)$ for all $x, y \in \mathbf{R}$.
e) For each $\alpha \in \mathbf{R}$ define $e^\alpha = E(\alpha)$. Let $x > 0$ and define $x^\alpha = e^{\alpha \log x} := E(\alpha L(x))$. Prove that $0 < x < y$ implies $x^\alpha < y^\alpha$ for $\alpha > 0$ and $x^\alpha > y^\alpha$ for $\alpha < 0$. Also prove that

$$x^{\alpha+\beta} = x^\alpha x^\beta, \qquad x^{-\alpha} = \frac{1}{x^\alpha}, \qquad \text{and} \qquad (x^\alpha)' = \alpha x^{\alpha-1}$$

for all $\alpha, \beta \in \mathbf{R}$ and $x > 0$.

5.3.9. Suppose that $f : [a, b] \to \mathbf{R}$ is continuously differentiable and 1–1 on $[a, b]$. Prove that

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a).$$

5.3.10. Suppose that ϕ is $\mathcal{C}^1$ on $[a, b]$ and f is integrable on $[c, d] := \phi[a, b]$. If ϕ' is never zero on $[a, b]$, prove that $f \circ \phi$ is integrable on $[a, b]$.

5.3.11. Let $q \in \mathbf{Q}$. Suppose that $a < b, 0 < c < d$, and that $f : [a, b] \to [c, d]$. If f is integrable on $[a, b]$, then prove that f^q is integrable on $[a, b]$.

5.3.12. For each $n \in \mathbf{N}$, define

$$a_n := \left(\frac{(2n)!}{n!n^n}\right)^{1/n}.$$

Prove that $a_n \to 4/e$.

5.4 IMPROPER RIEMANN INTEGRATION

To extend the Riemann integral to unbounded intervals or unbounded functions, we begin with an elementary observation.

5.37 Remark. *If f is integrable on $[a, b]$, then*

$$\int_a^b f(x)\,dx = \lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)\,dx \right).$$

Proof. By Theorem 5.26,

$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$. Thus

$$\int_a^b f(x)\,dx = F(b) - F(a) = \lim_{c \to a+} (\lim_{d \to b-} (F(d) - F(c)))$$
$$= \lim_{c \to a+} \left(\lim_{d \to b-} \int_c^d f(x)\,dx \right).$$

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This leads to the following generalization of the Riemann integral.