Math 125B, Winter 2015.

Homework 3 Solutions

5.3.0. (a) Yes. By Fundamental Theorem of Calculus and the chain rule: $F'(x) = f(g(x))g'(x)$. Both factors are nonnegative: $f \geq 0$ by the assumption, and $g' \geq 0$ as g is increasing.

In fact, we only need that $f \geq 0$, f is integrable, and g is increasing if we are willing to do a proof without derivatives. Namely, if $x_1 \leq x_2$, then $g(x_1) \leq g(x_2)$ and so

$$F(x_2) - F(x_1) = \int_{g(x_1)}^{g(x_2)} f(t)\,dt \geq 0$$

as $f \geq 0$.
(b) Yes. By quotient rule applied to $(f/g)'$ and linearity, the sum of two integrals equals $\int_a^x f' = f(x) - f(a)$, by fundamental theorem of calculus. Now use the assumption that $f(a) = 0$. (Another possibility is to use integration by parts.)
(c) Yes. The sum of integrals equals $f(b)g(b) - f(a)g(a)$ by integration by parts.

5.3.2. (b) Change the variable $x = t^{-1/2}$ to get

$$\frac{1}{2}\int_1^2 f(t)t^{-3/2}\,dt \leq \frac{1}{2}\int_1^2 f(t)\,dt = \frac{5}{2},$$

where the inequality holds because $t^{-3/2} \leq 1$ on $[1,2]$ and $f \geq 0$.

You should think *at least once* why the change of variables formalism as done in calculus follows from the theorem we proved. In this case, for example, the change of variables theorem is applied to $g(x) = x^{-2}$ and $f_1(x) = \frac{1}{2}f(x)x^{-3/2}$. Observe that g maps $[\sqrt{2}/2, 1]$ to $[1,2]$, and is differentiable on $(0,\infty)$, thus g' is continuous on $[\sqrt{2}/2, 1]$. Moreover, f_1 is continuous $[1,2]$. The change of variable theorem then implies:

$$\int_{\sqrt{2}/2}^1 f_1(g(x))g'(x)\,dx = \int_{g(\sqrt{2}/2)}^{g(1)} f_1(t)\,dt.$$

Now observe that $f_1(g(x))g'(x) = -f(1/x^2)$ and

$$\int_{g(\sqrt{2}/2)}^{g(1)} f_1(t)\,dt = -\frac{1}{2}\int_2^1 f(t)t^{-3/2}\,dt.$$

(c) Change the variable $t = x+1$ to get

$$\int_1^2 (t-1)^2 f(t)\,dt = 9 - 2\cdot 6 + 5 = 2.$$

5.3.4. (b) The integral equals

$$\int_0^1 \left(e^x f'(x) + (e^x)' f(x)\right)\,dx = e^1 f(1) - e^0 f(0),$$

by integration by parts, which is zero by the assumption.

5.3.6. Differentiate with respect to c to get, for every c,

$$\alpha f(c) - \beta f(c) = (\alpha - \beta) f(c) = 0,$$

and thus $f(c) = 0$.