To show that $\sin x/x$ is not absolutely integrable on $[1, \infty)$, notice that

$$\int_1^{n\pi} \frac{|\sin x|}{x}\,dx \geq \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx$$
$$\geq \sum_{k=2}^{n} \frac{1}{k\pi} \int_{(k-1)\pi}^{k\pi} |\sin x|\,dx$$
$$= \sum_{k=2}^{n} \frac{2}{k\pi} = \frac{2}{\pi} \sum_{k=2}^{n} \frac{1}{k}$$

for each $n \in \mathbf{N}\, n \geq 2$. Since

$$\sum_{k=2}^{n} \frac{1}{k} \geq \sum_{k=2}^{n} \int_{k}^{k+1} \frac{1}{x}\,dx = \int_{2}^{n+1} \frac{1}{x}\,dx = \log(n+1) - \log 2 \to \infty$$

as $n \to \infty$, it follows from the Squeeze Theorem that

$$\lim_{n\to\infty} \int_1^{n\pi} \frac{|\sin x|}{x}\,dx = \infty.$$

Thus, $\sin x/x$ is not absolutely integrable on $[1, \infty)$. ∎

EXERCISES

5.4.0. Suppose that $a < b$. Decide which of the following statements are true and which are false. Prove the true ones and give counterexamples for the false ones.

a) If f is bounded on $[a, b]$, if g is absolutely integrable on (a, b), and if $|f(x)| \leq g(x)$ for all $x \in (a, b)$, then f is absolutely integrable on (a, b).
b) Suppose that h is absolutely integrable on (a, b). If f is continuous on (a, b), if g is continuous and never zero on $[a, b]$, and if $|f(x)| \leq h(x)$ for all $x \in [a, b]$, then f/g is absolutely integrable on (a, b).
c) If $f : (a, b) \to [0, \infty)$ is continuous and absolutely integrable on (a, b) for some $a, b \in \mathbf{R}$, then $\sqrt{f}$ is absolutely integrable on (a, b).
d) If f and g are absolutely integrable on (a, b), then $\max\{f, g\}$ and $\min\{f, g\}$ are both absolutely integrable on (a, b).

5.4.1. Evaluate the following improper integrals.

a) $$\int_1^{\infty} \frac{1+x}{x^3}\,dx$$

b) $$\int_{-\infty}^{0} x^2 e^{x^3}\,dx$$

c)
$$\int_0^{\pi/2} \frac{\cos x}{\sqrt[3]{\sin x}}\,dx$$

d)
$$\int_0^1 \log x\,dx$$

5.4.2. For each of the following, find all values of $p \in \mathbf{R}$ for which f is improperly integrable on I.

a) $f(x) = 1/x^p$, $I = (1, \infty)$
b) $f(x) = 1/x^p$, $I = (0, 1)$
c) $f(x) = 1/(x \log^p x)$, $I = (e, \infty)$
d) $f(x) = 1/(1 + x^p)$, $I = (0, \infty)$
e) $f(x) = \log^a x/x^p$, where $a > 0$ is fixed, and $I = (1, \infty)$

5.4.3. Let $p > 0$. Show that $\sin x/x^p$ is improperly integrable on $[1, \infty)$ and that $\cos x/\log^p x$ is improperly integrable on $[e, \infty)$.

5.4.4. Decide which of the following functions are improperly integrable on I.

a) $f(x) = \sin x$, $I = (0, \infty)$
b) $f(x) = 1/x^2$, $I = [-1, 1]$
c) $f(x) = x^{-1}\sin(x^{-1})$, $I = (1, \infty)$
d) $f(x) = \log(\sin x)$, $I = (0, 1)$
e) $f(x) = (1 - \cos x)/x^2$, $I = (0, \infty)$

5.4.5. Use the examples provided by Exercise 5.4.2b to show that the product of two improperly integrable functions might not be improperly integrable.

5.4.6. Suppose that f, g are nonnegative and locally integrable on $[a, b)$ and that

$$L := \lim_{x \to b-} \frac{f(x)}{g(x)}$$

exists as an extended real number.

a) Show that if $0 \le L < \infty$ and g is improperly integrable on $[a, b)$, then so is f.
b) Show that if $0 < L \le \infty$ and g is not improperly integrable on $[a, b)$, then neither is f.

5.4.7. a) Suppose that f is improperly integrable on $[0, \infty)$. Prove that if $L = \lim_{x\to\infty} f(x)$ exists, then $L = 0$.

b) Let

$$f(x) = \begin{cases} 1 & n \le x < n + 2^{-n},\ n \in \mathbf{N} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that f is improperly integrable on $[0, \infty)$ but $\lim_{x\to\infty} f(x)$ does not exist.

5.4.8. Prove that if f is absolutely integrable on $[1, \infty)$, then

$$\lim_{n\to\infty} \int_1^\infty f(x^n)\, dx = 0.$$

5.4.9. Assuming $e = \lim_{n\to\infty} \sum_{k=0}^n 1/k!$ (see Example 7.45), prove that

$$\lim_{n\to\infty} \left(\frac{1}{n!} \int_1^\infty x^n e^{-x}\, dx \right) = 1.$$

5.4.10. a) Prove that

$$\int_0^{\pi/2} e^{-a \sin x}\, dx \le \frac{2}{a}$$

for all $a > 0$.

b) What happens if $\cos x$ replaces $\sin x$?

*5.5 FUNCTIONS OF BOUNDED VARIATION

This section uses no material from any other enrichment section.

In this section we study functions which do not wiggle too much. These functions, which play a prominent role in the theory of Fourier series (see Sections *14.3 and *14.4) and probability theory, are important tools for theoretical as well as applied mathematics.

Let $\phi : [a, b] \to \mathbf{R}$. To measure how much ϕ wiggles on an interval $[a, b]$, set

$$V(\phi, P) = \sum_{j=1}^n |\phi(x_j) - \phi(x_{j-1})|$$

for each partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. The *variation* of ϕ is defined by

$$\operatorname{Var}(\phi) := \sup\{V(\phi, P) : P \text{ is a partition of } [a, b]\}. \tag{19}$$

5.50 Definition.

Let $[a, b]$ be a closed, nondegenerate interval and $\phi : [a, b] \to \mathbf{R}$. Then ϕ is said to be of *bounded variation* on $[a, b]$ if and only if $\operatorname{Var}(\phi) < \infty$.

The following three remarks show how the collection of functions of bounded variation is related to other collections of functions we have studied.

5.51 Remark. *If $\phi \in C^1[a, b]$, then ϕ is of bounded variation on $[a, b]$. However, there exist functions of bounded variation which are not continuously differentiable.*

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. By the Extreme Value Theorem, there is an $M > 0$ such that $|\phi'(x)| \le M$ for all $x \in [a, b]$. Therefore,