converges as $n \to \infty$ by hypothesis, it follows from (5) and $b - a < \infty$ that f_n converges uniformly on (a, b) as $n \to \infty$.

Fix $c \in (a, b)$. Define f, g on (a, b) by $f(x) := \lim_{n\to\infty} f_n(x)$ and $g(x) := \lim_{n\to\infty} g_n(x)$. We need to show that

$$f'(c) = \lim_{n\to\infty} f_n'(c). \tag{6}$$

Since each g_n is continuous at c, the claim implies g is continuous at c. Since $g_n(c) = f_n'(c)$, it follows that the right side of (6) can be written as

$$\lim_{n\to\infty} f_n'(c) = \lim_{n\to\infty} g_n(c) = g(c) = \lim_{x\to c} g(x).$$

On the other hand, if $x \neq c$ we have by definition that

$$\frac{f(x) - f(c)}{x - c} = \lim_{n\to\infty} \frac{f_n(x) - f_n(c)}{x - c} = \lim_{n\to\infty} g_n(x) = g(x).$$

Therefore, the left side of (6) also reduces to

$$f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to c} g(x).$$

This verifies (6), and the proof of the theorem is complete. ■

EXERCISES

7.1.1. a) Prove that $x/n \to 0$ uniformly, as $n \to \infty$, on any closed interval $[a, b]$.

b) Prove that $1/(nx) \to 0$ pointwise but not uniformly on $(0, 1)$ as $n \to \infty$.

7.1.2. Prove that the following limits exist and evaluate them.

a) $\displaystyle \lim_{n\to\infty} \int_1^3 \frac{nx^{99} + 5}{x^3 + nx^{66}}\, dx$

b) $\displaystyle \lim_{n\to\infty} \int_0^2 e^{x^2/n}\, dx$

c) $\displaystyle \lim_{n\to\infty} \int_0^3 \sqrt{\sin \frac{x}{n} + x + 1}\, dx$

7.1.3. A sequence of functions f_n is said to be *uniformly bounded* on a set E if and only if there is an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in E$ and all $n \in \mathbf{N}$.

Suppose that for each $n \in \mathbf{N}$, $f_n : E \to \mathbf{R}$ is bounded. If $f_n \to f$ uniformly on E, as $n \to \mathbf{N}$, prove that $\{f_n\}$ is uniformly bounded on E and f is a bounded function on E.

7.1.4. Let $[a, b]$ be a closed bounded interval, $f : [a, b] \to \mathbf{R}$ be bounded, and $g : [a, b] \to \mathbf{R}$ be continuous with $g(a) = g(b) = 0$. Let f_n be a