Cauchy for each \( x \in E \).

By the Mean Value

\[
f'_n(\xi) - f'_m(\xi).
\]

there is an \( N \in \mathbb{N} \)

for \( x = c \) because

that by the claim, \( f'_n(x) \to f'_n(x_0) = x_0 \). Since \( f'_n(x_0) \)

Section 7.1 Uniform Convergence of Sequences

converges as \( n \to \infty \) by hypothesis, it follows from (5) and \( b - a < \infty \) that \( f_n \) converges uniformly on \((a, b)\) as \( n \to \infty \).

Fix \( c \in (a, b) \). Define \( f, g \) on \((a, b)\) by \( f(x) := \lim_{n \to \infty} f_n(x) \) and \( g(x) := \lim_{n \to \infty} g_n(x) \). We need to show that

\[
f'(c) = \lim_{n \to \infty} f'_n(c).
\]

Since each \( g_n \) is continuous at \( c \), the claim implies \( g \) is continuous at \( c \). Since \( g_n(c) = f'_n(c) \), it follows that the right side of (6) can be written as

\[
\lim_{n \to \infty} f'_n(c) = \lim_{n \to \infty} g_n(c) = g(c) = \lim_{x \to c} g(x).
\]

On the other hand, if \( x \neq c \) we have by definition that

\[
\frac{f(x) - f(c)}{x - c} = \lim_{n \to \infty} \frac{f_n(x) - f_n(c)}{x - c} = \lim_{n \to \infty} g_n(x) = g(x).
\]

Therefore, the left side of (6) also reduces to

\[
f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} g(x).
\]

This verifies (6), and the proof of the theorem is complete.

EXERCISES

7.1. a) Prove that \( x/n \to 0 \) uniformly, as \( n \to \infty \), on any closed interval \([a, b]\).

b) Prove that \( 1/(nx) \to 0 \) pointwise but not uniformly on \((0, 1)\) as \( n \to \infty \).

7.1.2. Prove that the following limits exist and evaluate them.

a) \( \lim_{n \to \infty} \int_{1}^{3} \frac{nx^6 + 5}{x^3 + nx^6} \, dx \)

b) \( \lim_{n \to \infty} \int_{0}^{2} e^{x^2/n} \, dx \)

c) \( \lim_{n \to \infty} \int_{0}^{3} \sqrt{\sin x/n + x + 1} \, dx \)

7.1.3. A sequence of functions \( f_n \) is said to be \textit{uniformly bounded} on a set \( E \) if and only if there is an \( M > 0 \) such that \( |f_n(x)| \leq M \) for all \( x \in E \) and all \( n \in \mathbb{N} \).

Suppose that for each \( n \in \mathbb{N} \), \( f_n : E \to \mathbb{R} \) is bounded. If \( f_n \to f \) uniformly on \( E \), as \( n \to \infty \), prove that \( \{f_n\} \) is uniformly bounded on \( E \) and \( f \) is a bounded function on \( E \).

7.1.4. Let \([a, b]\) be a closed bounded interval, \( f : [a, b] \to \mathbb{R} \) be bounded, and \( g : [a, b] \to \mathbb{R} \) be continuous with \( g(a) = g(b) = 0 \). Let \( f_n \) be a