
Homework 4 Solutions

5.1.0. (c) Yes. First, $\sqrt{f}$ is Riemann integrable on any closed interval on which $f$ is integrable, by composition with continuous function. Then, for any number $y \geq 0$, $\sqrt{y} \leq y + 1$ (when $y \leq 1$, this is trivial, and when $y \geq 1$, $\sqrt{y} \leq y$). So $\sqrt{f} \leq f + 1$. As both $f$ and $1$ are absolutely integrable on $(a, b)$, so is $\sqrt{f}$.

(d) Yes. By noting that, for example, $\max(f, g)$ equals either $f$ or $g$, it is clear that $|\max(f, g)| \leq \max(|f|, |g|) \leq |f| + |g|$ and $|\min(f, g)| \leq \max(|f|, |g|) \leq |f| + |g|$. Absolute integrability then follows from the comparison theorems.

5.4.1 (c). By change of variable $t = \sin x$,

$$\lim_{a \to 0^+} \int_a^{\pi/2} \frac{\cos x}{(\sin x)^{1/3}} \, dx = \lim_{a \to 0^+} \int_0^1 t^{-1/3} \, dt = \lim_{a \to 0^+} \frac{3}{2} (1 - (\sin a)^{2/3}) = \frac{3}{2}.$$ 

5.4.2 (d). By comparison to $g(x) = x^{-p}$ (observe that $f(x)/g(x) \to 1$ as $x \to \infty$), the integral converges exactly when $p > 1$.

5.4.4 (e). As $1 - \cos x = x^2/2 + O(x^4)$, the function is bounded near 0 and we only need to show that $\int_1^\infty f(x) \, dx$ is convergent. However, on this interval we may use the fact that $|f(x)| \leq 2x^{-2}$ and the fact that $\int_1^\infty x^{-2} \, dx$ is convergent, so $\int_1^\infty |f(x)| \, dx$ converges by comparison and then so does $\int_1^\infty f(x) \, dx$ by absolute convergence.

5.4.8. By change of variable $x = t^{1/n}$, one may rewrite

$$\left| \int_1^\infty f(x^n) \, dx \right| = \frac{1}{n} \left| \int_1^\infty f(t)t^{1/n-1} \, dt \right| \leq \frac{1}{n} \int_1^\infty |f(t)|t^{1/n-1} \, dt \leq \frac{1}{n} \int_1^\infty |f(t)| \, dt \to 0$$

as $n \to \infty$.

7.1.2 (a) Let $f_n(x) = \frac{nx^{39} + 5}{x^4 + nx^6}$ and $f(x) = x^{33}$. Then,

$$\sup_{x \in [1, 3]} |f_n(x) - f(x)| = \sup_{x \in [1, 3]} \frac{|5 - x^{36}|}{x^3 + nx^{66}} \leq \frac{5 + 3^{36}}{n},$$

which goes to 0 as $n \to \infty$. Thus $f_n \to f$ uniformly on $[1, 3]$ and then $\int_1^3 f_n \to \int_1^3 f = (3^{34} - 1)/34$.

(c) Let $f_n(x) = \sqrt{\sin(x/n)} + x + 1$ and $f(x) = \sqrt{x + 1}$. Then,

$$\sup_{x \in [0, 3]} |f_n(x) - f(x)| = \sup_{x \in [0, 3]} \frac{\sin(x/n) + x + 1 - (x + 1)}{\sqrt{\sin(x/n)} + x + 1 + \sqrt{x + 1}} \leq \sup_{x \in [0, 3]} \sin(x/n) \leq \sup_{x \in [0, 3]} x/n \leq 3/n,$$

goes to 0 as $n \to \infty$. Thus $f_n \to f$ uniformly on $[0, 3]$ and then $\int_0^3 f_n \to \int_0^3 f = \frac{14}{3}$. 
