Math 125B, Winter 2015.

Homework 4 Solutions

5.1.0. (c) Yes. First, $\sqrt{f}$ is Riemann integrable on any closed interval on which f is integrable, by composition with continuous function. Then, for any number $y \geq 0$, $\sqrt{y} \leq y+1$ (when $y \leq 1$, this is trivial, and when $y \geq 1$, $\sqrt{y} \leq y$). So $\sqrt{f} \leq f+1$. As both f and 1 are absolutely integrable on (a,b), so is $\sqrt{f}$.
(d) Yes. By noting that, for example, $\max(f,g)$ equals either f or g, it is clear that $|\max(f,g)| \leq \max(|f|,|g|) \leq |f|+|g|$ and $|\min(f,g)| \leq \max(|f|,|g|) \leq |f|+|g|$. Absolute integrability then follows from the comparison theorems.

5.4.1 (c). By change of variable $t = \sin x$,

$$\lim_{a\to 0+} \int_a^{\pi/2} \frac{\cos x}{(\sin x)^{1/3}}\, dx = \lim_{a\to 0+} \int_{\sin a}^{1} t^{-1/3}\, dx = \lim_{a\to 0+} \frac{3}{2}(1-(\sin a)^{2/3}) = \frac{3}{2}.$$

5.4.2 (d). By comparison to $g(x) = x^{-p}$ (observe that $f(x)/g(x) \to 1$ as $x \to \infty$), the integral converges exactly when $p > 1$.

5.4.4 (e). As $1 - \cos x = x^2/2 + \mathcal{O}(x^4)$, the function is bounded near 0 and we only need to show that $\int_1^\infty f(x)\, dx$ is convergent. However, on this interval we may use the fact that $|f(x)| \leq 2x^{-2}$ and the fact that $\int_1^\infty x^{-2}\, dx$ is convergent, so $\int_1^\infty |f(x)|\, dx$ converges by comparison and then so does $\int_1^\infty f(x)\, dx$ by absolute convergence.

5.4.8. By change of variable $x = t^{1/n}$, one may rewrite

$$\left|\int_1^\infty f(x^n)\, dx\right| = \frac{1}{n}\left|\int_1^\infty f(t) t^{1/n-1}\, dt\right| \leq \frac{1}{n}\int_1^\infty |f(t)| t^{1/n-1}\, dt \leq \frac{1}{n}\int_1^\infty |f(t)|\, dt \to 0$$

as $n \to \infty$.

7.1.2 (a) Let $f_n(x) = \frac{nx^{99}+5}{x^3+nx^{66}}$ and $f(x) = x^{33}$. Then,

$$\sup_{x\in[1,3]} |f_n(x) - f(x)| = \sup_{x\in[1,3]} \frac{|5-x^{36}|}{x^3+nx^{66}} \leq \frac{5+3^{36}}{n},$$

which goes to 0 as $n \to \infty$. Thus $f_n \to f$ uniformly on $[1,3]$ and then $\int_1^3 f_n \to \int_1^3 f = (3^{34}-1)/34$.
(c)Let $f_n(x) = \sqrt{\sin(x/n)+x+1}$ and $f(x) = \sqrt{x+1}$. Then,

$$\sup_{x\in[0,3]} |f_n(x) - f(x)| = \sup_{x\in[0,3]} \frac{|\sin(x/n)+x+1-(x+1)|}{\sqrt{\sin(x/n)+x+1}+\sqrt{x+1}} \leq \sup_{x\in[0,3]} \sin(x/n) \leq \sup_{x\in[0,3]} x/n \leq 3/n,$$

goes to 0 as $n \to \infty$. Thus $f_n \to f$ uniformly on $[0,3]$ and then $\int_0^3 f_n \to \int_0^3 f = \frac{14}{3}$.