c) $$\mathbf{f}(x, y) = \left( \frac{x^4 + y^4}{x^2 + y^2}, \frac{\sqrt{|xy|}}{\sqrt[3]{x^2 + y^2}} \right), \qquad (a, b) = (0, 0)$$

d) $$\mathbf{f}(x, y) = \left( \frac{x^2 - 1}{y^2 + 1}, \frac{x^2y - 2xy + y - (x - 1)^2}{x^2 + y^2 - 2x - 2y + 2} \right), \qquad (a, b) = (1, 1)$$

**9.3.2.** Compute the iterated limits at (0,0) of each of the following functions. Determine which of these functions has a limit as $(x, y) \to (0, 0)$ in $\mathbf{R}^2$, and prove that the limit exists.

a) $$f(x, y) = \frac{\sin x \sin y}{x^2 + y^2}$$

b) $$f(x, y) = \frac{x^2 + y^4}{x^2 + 2y^4}$$

c) $$f(x, y) = \frac{x - y}{(x^2 + y^2)^\alpha}, \qquad \alpha < \frac{1}{2}$$

**9.3.3.** Prove that each of the following functions has a limit as $(x, y) \to (0, 0)$.

a) $$f(x, y) = \frac{x^3 - y^3}{x^2 + y^2}, \qquad (x, y) \neq (0, 0)$$

b) $$f(x, y) = \frac{|x|^\alpha y^4}{x^2 + y^4}, \qquad (x, y) \neq (0, 0),$$

where $\alpha$ is ANY positive number.

**9.3.4.** A *polynomial* on $\mathbf{R}^n$ of *degree* $N$ is a function of the form

$$P(x_1, x_2, \dots, x_n) = \sum_{j_1 = 0}^{N_1} \cdots \sum_{j_n = 0}^{N_n} a_{j_1, \dots, j_n} x_1^{j_1} \dots x_n^{j_n},$$

where $a_{j_1, \dots, j_n}$ are scalars, $N_1, \dots, N_n$ are nonnegative integers, and $N = N_1 + N_2 + \cdots + N_n$. Prove that if $P$ is a polynomial on $\mathbf{R}^n$ and $\mathbf{a} \in \mathbf{R}^n$, then $\lim_{\mathbf{x} \to \mathbf{a}} P(\mathbf{x}) = P(\mathbf{a})$.

**9.3.5.** Suppose that $\mathbf{a} \in \mathbf{R}^n$, that $\mathbf{L} \in \mathbf{R}^m$, and that $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^m$. Prove that if $\mathbf{f}(\mathbf{x}) \to \mathbf{L}$ as $\mathbf{x} \to \mathbf{a}$, then there is an open set $V$ containing $\mathbf{a}$ and a constant $M > 0$ such that $\|\mathbf{f}(\mathbf{x})\| \le M$ for all $\mathbf{x} \in V$.

**9.3.6.** Suppose that $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{R}^n$, that $f_j : \mathbf{R} \to \mathbf{R}$ for $j = 1, 2, \dots, n$, and that $g(x_1, x_2, \dots, x_n) := f_1(x_1) \cdots f_n(x_n)$.

a) Prove that if $f_j(t) \to f_j(a_j)$ as $t \to a_j$, for each $j = 1, \dots, n$, then $g(\mathbf{x}) \to f_1(a_1) \cdots f_n(a_n)$ as $\mathbf{x} \to \mathbf{a}$.