of the interval $[0, 1]$, we have by Theorems 8.30 and 9.30 that E is connected. On the other hand, since $E \subset R$ by the definition of R, it is easy to check that $U_0 := U \cap E$ and $V_0 := V \cap E$ are nonempty sets which are relatively open in E and satisfy $E = U_0 \cup V_0$ and $U_0 \cap V_0 = \emptyset$. It follows that E is not connected, a contradiction. ∎

EXERCISES

9.4.1. Define f and g on $\mathbf{R}$ by $f(x) = \sin x$ and $g(x) = x/|x|$ if $x \neq 0$ and $g(0) = 0$.

a) Find $f(E)$ and $g(E)$ for $E = (0, \pi)$, $E = [0, \pi]$, $E = (-1, 1)$, and $E = [-1, 1]$. Compare your answers with what Theorems 9.26, 9.29, and 9.30 predict. Explain any differences you notice.

b) Find $f^{-1}(E)$ and $g^{-1}(E)$ for $E = (0, 1)$, $E = [0, 1]$, $E = (-1, 1)$, and $E = [-1, 1]$. Compare your answers with what Theorems 9.26, 9.29, and 9.30 predict. Explain any differences you notice.

9.4.2. Define f on $[0, \infty)$ and g on $\mathbf{R}$ by $f(x) = \sqrt{x}$ and $g(x) = 1/x$ if $x \neq 0$ and $g(0) = 0$.

a) Find $f(E)$ and $g(E)$ for $E = (0, 1)$, $E = [0, 1)$, and $E = [0, 1]$. Compare your answers with what Theorems 9.26, 9.29, and 9.30 predict. Explain any differences you notice.

b) Find $f^{-1}(E)$ and $g^{-1}(E)$ for $E = (-1, 1)$ and $E = [-1, 1]$. Compare your answers with what Theorems 9.26, 9.29, and 9.30 predict. Explain any differences you notice.

9.4.3. **This exercise is used in this section and in Chapter 11.** Suppose that A is open in $\mathbf{R}^n$ and $\mathbf{f} : A \to \mathbf{R}^m$. Prove that $\mathbf{f}$ is continuous on A if and only if $\mathbf{f}^{-1}(V)$ is open in $\mathbf{R}^n$ for every open subset V of $\mathbf{R}^m$.

9.4.4. Suppose that A is closed in $\mathbf{R}^n$ and $\mathbf{f} : A \to \mathbf{R}^m$. Prove that $\mathbf{f}$ is continuous on A if and only if $\mathbf{f}^{-1}(E)$ is closed in $\mathbf{R}^n$ for every closed subset E of $\mathbf{R}^m$.

9.4.5. Suppose that $E \subseteq \mathbf{R}^n$ and that $\mathbf{f} : E \to \mathbf{R}^m$.

a) Prove that $\mathbf{f}$ is continuous on E if and only if $\mathbf{f}^{-1}(B)$ is relatively closed in E for every closed subset B of $\mathbf{R}^m$.

b) Suppose that $\mathbf{f}$ is continuous on E. Prove that if V is relatively open in $\mathbf{f}(E)$, then $\mathbf{f}^{-1}(V)$ is relatively open in E, and if B is relatively closed in $\mathbf{f}(E)$, then $\mathbf{f}^{-1}(B)$ is relatively closed in E.

9.4.6. Prove that

$$f(x, y) = \begin{cases} e^{-1/|x-y|} & x \neq y \\ 0 & x = y \end{cases}$$

is continuous on $\mathbf{R}^2$.

***9.4.7.** **This exercise is used in Section *9.6.** Let H be a nonempty, closed, bounded subset of $\mathbf{R}^n$.

a) Suppose that $\mathbf{f} : H \to \mathbf{R}^m$ is continuous. Prove that

$$\|\mathbf{f}\|_H := \sup_{x \in H} \|\mathbf{f}(\mathbf{x})\|$$

is finite and there exists an $\mathbf{x}_0 \in H$ such that $\|\mathbf{f}(\mathbf{x}_0)\| = \|\mathbf{f}\|_H$.

b) A sequence of functions $\mathbf{f}_k : H \to \mathbf{R}^m$ is said to converge uniformly on H to a function $\mathbf{f} : H \to \mathbf{R}^m$ if and only if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$k \geq N \quad \text{and} \quad \mathbf{x} \in H \quad \text{imply} \quad \|\mathbf{f}_k(\mathbf{x}) - \mathbf{f}(\mathbf{x})\| < \varepsilon.$$

Show that $\|\mathbf{f}_k - \mathbf{f}\|_H \to 0$ as $k \to \infty$ if and only if $\mathbf{f}_k \to \mathbf{f}$ uniformly on H as $k \to \infty$.

c) Prove that a sequence of functions $\mathbf{f}_k$ converges uniformly on H if and only if for every $\varepsilon > 0$ there is an $N \in \mathbf{N}$ such that

$$k, j \geq N \quad \text{implies} \quad \|\mathbf{f}_k - \mathbf{f}_j\|_H < \varepsilon.$$

9.4.8. Let $E \subset \mathbf{R}^n$ and suppose that D is *dense* in E (i.e., that $D \subseteq E$ and $\overline{D} = E$). If $\mathbf{f} : D \to \mathbf{R}^m$ is uniformly continuous on D, prove that $\mathbf{f}$ has a continuous extension to E; that is, prove that there is a continuous function $\mathbf{g} : E \to \mathbf{R}^m$ such that $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in D$.

9.4.9. [INTERMEDIATE VALUE THEOREM]. Let E be a connected subset of $\mathbf{R}^n$. If $f : E \to \mathbf{R}$ is continuous, $f(\mathbf{a}) \neq f(\mathbf{b})$ for some $\mathbf{a}, \mathbf{b} \in E$, and y is a number which lies between $f(\mathbf{a})$ and $f(\mathbf{b})$, then prove that there is an $\mathbf{x} \in E$ such that $f(\mathbf{x}) = y$. (You may use Theorem 8.30.)

***9.4.10. This exercise is used to prove *Corollary 11.35.**

a) A set $E \subseteq \mathbf{R}^n$ is said to be *polygonally connected* if and only if any two points $\mathbf{a}, \mathbf{b} \in E$ can be connected by a polygonal path in E; that is, there exist points $\mathbf{x}_k \in E$, $k = 1, \dots, N$, such that $\mathbf{x}_0 = \mathbf{a}$, $\mathbf{x}_N = \mathbf{b}$ and $L(\mathbf{x}_{k-1}; \mathbf{x}_k) \subseteq E$ for $k = 1, \dots, N$. Prove that every polygonally connected set in $\mathbf{R}^n$ is connected.

b) Let $E \subseteq \mathbf{R}^n$ be open and $\mathbf{x}_0 \in E$. Let U be the set of points $\mathbf{x} \in E$ which can be polygonally connected in E to $\mathbf{x}_0$. Prove that U is open.

c) Prove that every open connected set in $\mathbf{R}^n$ is polygonally connected.

*9.5 COMPACT SETS

This section requires no material from any other enrichment section.

In this section we give a more complete description of compact sets. Most of the results we state are trivial to prove by appealing to the hard part of Heine–Borel Theorem, specifically, that closed and bounded subsets of a Euclidean space are compact. Since this powerful result does not hold in some non-Euclidean spaces, our proofs will appeal only to the basic definition of compact sets and, hence, avoid using the Heine–Borel Theorem.