Math 125B, Winter 2015.

Homework 5 Solutions

9.3.2. (a) In all such problems, write $(x, y) = r(a, b)$, where $a^2 + b^2 = 1$. Observe that $|a|, |b| \le 1$ and one of $|a|$, $|b|$ is at least $1/\sqrt{2} > 1/2$.

If a and b are fixed (that is, the point (x, y) lies on a fixed line through the origin), then

$$f(x, y) = \frac{\sin(ra)\sin(rb)}{r^2} \to ab,$$

and so the limit depends on a and b. Thus $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.
(d) Now

$$f(x, y) = \frac{a^2 + b^4 r^2}{a^2 + 2b^4 r^2}.$$

Assume a, b are fixed. Then the above expression approaches 1 as $r \to 0$ as soon as $a \neq 0$. If $a = 0$, however, this equals $1/2$ for all r. Thus the limit does not exist.

9.3.3 (a). Here we have

$$|f(x, y)| = r|a^3 - b^3| \le 2r$$

and the last bound is independent of a and b, and goes to 0 as $r \to 0$.

9.3.5. By defintion of the limit, there is a $\delta > 0$ so that $||f(x) - L|| < 1$ for all $x \in B_\delta(a)$. Let $V = B_\delta(a)$, which is an open set. On V,

$$||f(x)|| \le ||f(x) - L|| + ||L|| \le 1 + ||L||,$$

so we can take $M = ||L|| + 1$.

9.4.6. We only need to prove continuity at a point (x_0, x_0), $x_0 \in \mathbb{R}$. Assume that a sequence of points $(x_n, y_n) \to (x_0, x_0)$. Then both $x_n \to x_0$ and $y_n \to x_0$, and so $|x_n - y_n| \to 0$. Then $1/|x_n - y_n| \to \infty$ and $f(x_n, y_n) = e^{-1/|x_n - y_n|} \to 0 = f(x_0, y_0)$. Thus f is continuous at (x_0, y_0).

9.4.8. Assume that f is uniformly continuous. Fix an $x \in E$ and take a sequence $x_k \in D$ so that $x_k \to x$ as $k \to \infty$.

We first prove that $f(x_k)$ is a Cauchy sequence. Fix an $\epsilon > 0$. Then there exists a $\delta > 0$ so that $||x - y|| < \delta$ implies $||f(x) - f(y)|| < \epsilon$. Then there exist a K so that $k, \ell \ge K$ implies that $||x_k - x_\ell|| < \delta$ and thus that $||f(x_k) - f(x_\ell)|| < \epsilon$.

Therefore $f(x_k)$ converges to, say, L. We next show that L is independent of the choice of the sequence (x_k), provided it converges to x. Assume (x'_k) is another such sequence, with $L' = \lim f(x'_k)$. Then $\lim_k ||x_k - x'_k|| = 0$ and, as f is uniformly continuous, $||L - L'|| = \lim_k ||f(x_k) - f(x'_k)|| = 0$. Thus $L = L'$. We may therefore define $g(x) = \lim f(x_k)$, where $x_k \in D$ is an arbitrary sequence that converges to x. Observe that this means that $g(x) = f(x)$ if $x \in E$, as in this case the we can take $x_k = x$ for all k.

Now we show that g is uniformly continuous. Pick an $\epsilon > 0$. Find a $\delta > 0$ so that $||f(y') - f(y)|| < \epsilon/3$ for all $y, y' \in D$ with $||y' - y|| < \delta$. We claim that if $x, x' \in E$ with $||x - x'|| < \delta/3$, then $||g(x') - g(x)|| < \epsilon$.

Find a $y \in D$ so that $||x - y|| < \delta/3$ and $||f(y) - g(x)|| < \epsilon/3$. (We can do so because we there is a sequence of points in $x_k \in D$ with $x_k \to x$ and $f(x_k) \to g(x)$.) Similarly, find a $y' \in D$ so that $||x' - y'|| < \delta/3$ and $||f(y') - g(x')|| < \epsilon/3$. Then $||y' - y|| \leq ||y' - x'|| + ||x' - x|| + ||x - y|| < \delta$ and therefore $||f(y') - f(y)|| < \epsilon/3$. Therefore

$$||g(x') - g(x)|| \leq ||g(x') - f(y')|| + ||f(y') - f(y)|| + ||f(y) - f(x)|| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$