for all $y \in [c, d]$ which satisfy $|y - y_0| < \delta$. Then

$$\begin{aligned}|F(y) - F(y_0)| &\le \left|F(y) - \int_A^B f(x, y)\,dx\right| + \left|\int_A^B (f(x, y) - f(x, y_0))\,dx\right| \\ &\quad + \left|F(y_0) - \int_A^B f(x, y_0)\,dx\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for all $y \in [c, d]$ which satisfy $|y - y_0| < \delta$. ■

The proof of Theorem 11.5 can be modified to prove the following result.

***11.9 Theorem.** *Suppose that $a < b$ are extended real numbers, that $c < d$ are finite real numbers, that $f : (a, b) \times [c, d] \to \mathbf{R}$ is continuous, and that the improper integral*

$$F(y) = \int_a^b f(x, y)\,dx$$

exists for all $y \in [c, d]$. If $f_y(x, y)$ exists and is continuous on $(a, b) \times [c, d]$ and if

$$\phi(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\,dx$$

converges uniformly on $[c, d]$, then F is differentiable on $[c, d]$ and $F'(y) = \phi(y)$; that is,

$$\frac{d}{dy}\int_a^b f(x, y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x, y)\,dx$$

for all $y \in [c, d]$.

For a result about interchanging two partial integrals, see Theorem 12.31 and Exercise 12.3.10.

EXERCISES

11.1.1. Compute all mixed second-order partial derivatives of each of the following functions and verify that the mixed partial derivatives are equal.

a) $f(x, y) = xe^y$ b) $f(x, y) = \cos(xy)$ c) $f(x, y) = \dfrac{x + y}{x^2 + 1}$

11.1.2. For each of the following functions, compute f_x and determine where it is continuous.

a) $$f(x,y) = \begin{cases} \dfrac{x^4+y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

b) $$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{\sqrt[3]{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

11.1.3. Suppose that $r > 0$, that $\mathbf{a} \in \mathbf{R}^n$, and that $\mathbf{f} : B_r(\mathbf{a}) \to \mathbf{R}^m$. If all first-order partial derivatives of $\mathbf{f}$ exist on $B_r(\mathbf{a})$ and satisfy $\mathbf{f}_{x_j}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in B_r(\mathbf{a})$ and all $j = 1, 2, \dots, n$, prove that $\mathbf{f}$ has only one value on $B_r(\mathbf{a})$.

11.1.4. Suppose that $H = [a,b] \times [c,d]$ is a rectangle, that $f : H \to \mathbf{R}$ is continuous, and that $g : [a,b] \to \mathbf{R}$ is integrable. Prove that

$$F(y) = \int_a^b g(x) f(x,y)\, dx$$

is uniformly continuous on $[c,d]$.

11.1.5. Evaluate each of the following expressions.

a) $$\lim_{y \to 0} \int_0^1 e^{x^3 y^2 + x}\, dx$$

b) $$\frac{d}{dy} \int_0^1 \sin(e^x y - y^3 + \pi - e^x)\, dx \quad \text{at } y = 1$$

c) $$\frac{\partial}{\partial x} \int_1^3 \sqrt{x^3 + y^3 + z^3 - 2}\, dz \quad \text{at } (x,y) = (1,1)$$

11.1.6. Suppose that f is a continuous real function.

a) If $\int_0^1 f(x)\, dx = 1$, find the exact value of

$$\lim_{y \to 0} \int_0^2 f(|x-1|) e^{x^2 y + x y^2}\, dx.$$

b) If f is $\mathcal{C}^1$ on $\mathbf{R}$ and $\int_0^\pi f'(x) \sin x dx = e$, find the exact value of

$$e + \lim_{y \to 0} \int_0^\pi f(x) \cos(y^5 + \sqrt[3]{y} + x)\, dx.$$

c) If $\int_0^1 f(\sqrt{x}) e^x dx = 6$, find the exact value of

$$\frac{d}{dx} \int_0^1 f(y) e^{xy + y^2}\, dy \quad \text{at } x = 0.$$