for all \( y \in [c, d] \) which satisfy \(|y - y_0| < \delta\). Then

\[
|F(y) - F(y_0)| \leq |F(y) - \int_A^B f(x, y) \, dx| + \left| \int_A^B (f(x, y) - f(x, y_0)) \, dx \right|
\]
\[
+ |F(y_0) - \int_A^B f(x, y_0) \, dx| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

for all \( y \in [c, d] \) which satisfy \(|y - y_0| < \delta\).

The proof of Theorem 11.5 can be modified to prove the following result.

**11.9 Theorem.** Suppose that \(a < b\) are extended real numbers, that \(c < d\) are finite real numbers, that \( f : (a, b) \times [c, d] \to \mathbb{R} \) is continuous, and that the improper integral

\[
F(y) = \int_a^b f(x, y) \, dx
\]

exists for all \( y \in [c, d] \). If \( f_y(x, y) \) exists and is continuous on \((a, b) \times [c, d]\) and if

\[
\phi(y) = \int_a^b \frac{\partial f}{\partial y}(x, y) \, dx
\]

converges uniformly on \([c, d]\), then \( F \) is differentiable on \([c, d]\) and \( F'(y) = \phi(y) \); that is,

\[
\frac{d}{dy} \int_a^b f(x, y) \, dx = \int_a^b \frac{\partial f}{\partial y}(x, y) \, dx
\]

for all \( y \in [c, d] \).

For a result about interchanging two partial integrals, see Theorem 12.31 and Exercise 12.3.10.

**EXERCISES**

**11.1.** Compute all mixed second-order partial derivatives of each of the following functions and verify that the mixed partial derivatives are equal.

\[
a) \ f(x, y) = xe^y \quad b) \ f(x, y) = \cos(xy) \quad c) \ f(x, y) = \frac{x + y}{x^2 + 1}
\]

**11.2.** For each of the following functions, compute \( f_x \) and determine where it is continuous.
a) \[ f(x, y) = \begin{cases} \frac{x^4 + y^4}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]

b) \[ f(x, y) = \begin{cases} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]

11.1.3. Suppose that \( r > 0 \), that \( a \in \mathbb{R}^n \), and that \( f : B_r(a) \to \mathbb{R}^n \). If all first-order partial derivatives of \( f \) exist on \( B_r(a) \) and satisfy \( f_{ij}(x) = 0 \) for all \( x \in B_r(a) \) and all \( j = 1, 2, \ldots, n \), prove that \( f \) has only one value on \( B_r(a) \).

11.1.4. Suppose that \( H = [a, b] \times [c, d] \) is a rectangle, that \( f : H \to \mathbb{R} \) is continuous, and that \( g : [a, b] \to \mathbb{R} \) is integrable. Prove that

\[ F(y) = \int_a^b g(x) f(x, y) \, dx \]

is uniformly continuous on \([c, d] \).

11.1.5. Evaluate each of the following expressions.

a) \[ \lim_{y \to 0} \int_0^1 e^{x^2 y^2 + x} \, dx \]

b) \[ \frac{d}{dy} \int_0^1 \sin(e^x y - y^3 + \pi - e^x) \, dx \] at \( y = 1 \)

c) \[ \frac{\partial}{\partial x} \int_1^3 \sqrt{x^3 + y^3 + z^3 - 2} \, dz \] at \( (x, y) = (1, 1) \)

11.1.6. Suppose that \( f \) is a continuous real function.

a) If \( \int_0^1 f(x) \, dx = 1 \), find the exact value of

\[ \lim_{y \to 0} \int_0^2 f(|x - 1|) e^{x^2 y + xy^2} \, dx. \]

b) If \( f \) is \( C^1 \) on \( \mathbb{R} \) and \( \int_0^\pi f(x) \sin x \, dx = e \), find the exact value of

\[ e + \lim_{y \to 0} \int_0^\pi f(x) \cos(y^5 + \sqrt{y} + x) \, dx. \]

c) If \( \int_0^1 f(\sqrt{x}) e^x \, dx = 6 \), find the exact value of

\[ \frac{d}{dx} \int_0^1 f(y) e^{xy + y^2} \, dy \] at \( x = 0. \)