Math 125B, Winter 2015.

Homework 6 Solutions

11.1.2. (a) For $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{2x(x^4 + 2x^2y^2 - y^4)}{(x^2+y^2)^2},$$

while $f(x,0) = x^2$ implies $f_x(0,0) = 0$. If $(x,y) = r(a,b)$ with $a^2 + b^2 = 1$, we have

$$|f_x(x,y)| = r \cdot |2a(a^4 + 2a^2b^2 - b^4)| \leq r \cdot 8,$$

as $|a| \leq 1$ and $|b| \leq 1$. The bound above is independent of (a,b) and goes to 0 as $r \to 0$. Therefore f_x is continuous everywhere on $\mathbb{R}^2$.

(d) Now, for $(x,y) \neq (0,0)$,

$$f_x(x,y) = \frac{4x(x^2+2y^2)}{3(x^2+y^2)^{4/3}}$$

and $f(x,0) = x^{4/3}$, so again $f_x(0,0) = 0$. Using the same representation as in (a),

$$|f_x(x,y)| = \frac{4}{3}r^{1/3} \cdot |a(a^2+2b^2)| \leq 4r^{1/3},$$

so again we have a bound indepenent of r that goes to 0 as $r \to 0$. Therefore f_x is continuous everywhere on $\mathbb{R}^2$.

11.1.3. We may assume f is a scalar function, that is, that $m = 1$, as otherwise we argue for each coordinate function separately.

First we show that if $x, y \in B_r(a)$ differ in only one coordinate, then $f(x) = f(y)$. To show this, assume that, for some $k \leq n$, $x_k < y_k$ and $x_i = y_i$ for $i \neq k$, that is, x and y lie on a line parallel to one of the axes. Define the vector

$$v(t) = (x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n).$$

The first thing to show is that $v(t) \in B_r(a)$ for every $t \in [x_k, y_k]$. (One can easily prove this this by using the fact that $B_r(a)$ is convex, but we present a direct argument.) For such t, $x_k - a_k \leq t - a_k \leq y_k - a_k$ and so either $|t - a_k| \leq |y_k - a_k|$ (if $t - a_k \geq 0$) or $|t - a_k| \leq |x_k - a_k|$ (if $t - a_k \leq 0$). In the first case, $||v(t) - a|| \leq ||x - a|| \leq r$ and in the second $||v(t) - a|| \leq ||y - a|| \leq r$.

We now define the function $g : [x_k, y_k] \to \mathbb{R}$ by

$$g(t) = f(v(t)) = f(x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n).$$

Then g is differentiable and $g'(t) = \frac{\partial f}{\partial x_k}(v(t)) = 0$ and so g is constant; in particular $f(x) = g(x_k) = g(y_k) = f(y)$.

Assume now $x \in B_r(a)$ is arbitrary. We will prove that $f(x) = f(a)$ by connecting a and x with lines parallel to the coordinate axes. For $k = 0, 1, \dots, n$, consider the vector v_k which has first k

coordinates equal to those of x and the remaining coordinates equal to those of a. Thus $v_0 = a$ and $v_n = x$; also,

$$||v_k - a||^2 = (x_1 - a_1)^2 + \ldots + (x_k - a_k)^2 \leq (x_1 - a_1)^2 + \ldots + (x_n - a_n)^2 = ||x - a||^2 < r^2,$$

so $v_k \in B_r(a)$ for all k. Moreover, for each $1 \leq k \leq n$, v_{k-1} and v_k differ only in the kth coordinate, and so $f(v_{k-1}) = f(v_k)$. Therefore $f(v_0) = f(v_n)$, that is, $f(a) = f(x)$.