Section 11.2  The Definition of Differentiability

Proof. If \((x, y) \neq (0, 0)\), then we can use the one-dimensional Product Rule to verify that both \(f_x\) and \(f_y\) exist and are continuous, for example,

\[
f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \cos \frac{1}{\sqrt{x^2 + y^2}} + 2x \sin \frac{1}{\sqrt{x^2 + y^2}}.
\]

Thus \(f\) is differentiable on \(\mathbb{R}^2 \setminus \{(0, 0)\}\). Since \(f_x(x, 0)\) has no limit as \(x \to 0\), the partial derivative \(f_x\) is not continuous at \((0, 0)\). A similar statement holds for \(f_y\). Thus to check differentiability at \((0, 0)\) we must return to the definition.

First, we compute the partial derivatives at \((0, 0)\). By definition,

\[
f_x(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \to 0} t \sin \frac{1}{t} = 0.
\]

and similarly, \(f_y(0, 0) = 0\). Thus, both first partials exist at \((0, 0)\) and \(\nabla f(0, 0) = (0, 0)\).

To prove that \(f\) is differentiable at \((0, 0)\), we must verify (4) for \(a = (0, 0)\). But it is clear that

\[
\frac{f(h, k) - f(0, 0) - \nabla f(0, 0) \cdot (h, k)}{\| (h, k) \|} = \frac{\sqrt{h^2 + k^2} \sin \frac{1}{\sqrt{h^2 + k^2}}}{\sqrt{h^2 + k^2}} \to 0
\]

as \((h, k) \to (0, 0)\). Thus \(f\) is differentiable at \((0, 0)\). \(\square\)

EXERCISES

11.2.1. Suppose, for \(j = 1, 2, \ldots, n\), that \(f_j\) are real functions continuously differentiable on the interval \((-1, 1)\). Prove that

\[
g(x) := f_1(x_1) \cdots f_n(x_n)
\]

is differentiable on the cube \((-1, 1) \times (-1, 1) \times \cdots \times (-1, 1)\).

11.2.2. Suppose that \(f, g : \mathbb{R} \to \mathbb{R}^n\) are differentiable at \(a\) and there is a \(\delta > 0\) such that \(g(x) \neq 0\) for all \(0 < |x - a| < \delta\). If \(f(a) = g(a) = 0\) and \(Dg(a) \neq 0\), prove that

\[
\lim_{x \to a} \frac{\|f(x)\|}{\|g(x)\|} = \frac{\|Df(a)\|}{\|Dg(a)\|}.
\]

11.2.3. Prove that \(f(x, y) = \sqrt{|xy|}\) is not differentiable at \((0, 0)\).

11.2.4. Prove that

\[
f(x, y) = \begin{cases} \frac{x^2 + y^2}{\sin \sqrt{x^2 + y^2}} & 0 < \|(x, y)\| < \pi \\ 0 & (x, y) = (0, 0) \end{cases}
\]

is not differentiable at \((0, 0)\).
11.2.5. Prove that
\[ f(x, y) = \begin{cases} \frac{x^4 + y^4}{(x^2 + y^2)^\alpha} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]
is differentiable on \( \mathbb{R}^2 \) for all \( \alpha < 3/2 \).

11.2.6. Prove that if \( \alpha > 1/2 \), then
\[ f(x, y) = \begin{cases} |xy|^\alpha \log(x^2 + y^2) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]
is differentiable at \((0, 0)\).

11.2.7. Prove that
\[ f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]
is continuous on \( \mathbb{R}^2 \) and has first-order partial derivatives everywhere on \( \mathbb{R}^2 \), but \( f \) is not differentiable at \((0, 0)\).

11.2.8. This exercise is used several times in this chapter and the next. Suppose that \( T \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m) \). Prove that \( T \) is differentiable everywhere on \( \mathbb{R}^n \) with
\[ DT(a) = T \quad \text{for} \quad a \in \mathbb{R}^n. \]

11.2.9. Let \( r > 0 \), \( f : B_r(0) \to \mathbb{R} \), and suppose that there exists an \( \alpha > 1 \) such that \( |f(x)| \leq \|x\|^\alpha \) for all \( x \in B_r(0) \). Prove that \( f \) is differentiable at \( 0 \). What happens to this result when \( \alpha = 1 \)?

11.2.10. Let \( V \) be open in \( \mathbb{R}^n \), \( a \in V \), and \( f : V \to \mathbb{R}^m \).

11.19 Definition.

If \( u \) is a unit vector in \( \mathbb{R}^n \) (i.e., \( \|u\| = 1 \)), then the directional derivative of \( f \) at \( a \) in the direction \( u \) is defined by
\[ D_u f(a) := \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t} \]
when this limit exists.

a) Prove that \( D_u f(a) \) exists for \( u = e_k \) if and only if \( f_{x_k}(a) \) exists, in which case
\[ D_{e_k} f(a) = \frac{\partial f}{\partial x_k}(a). \]

b) Show that if \( f \) has directional derivatives at \( a \) in all directions \( u \), then the first-order partial derivatives of \( f \) exist at \( a \). Use Example 11.11 to show that the converse of this statement is false.
c) Prove that the directional derivatives of

\[ f(x, y) = \begin{cases} 
\frac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\
0 & (x, y) = (0, 0)
\end{cases} \]

exist at (0, 0) in all directions \( \mathbf{u} \), but \( f \) is neither continuous nor differentiable at (0, 0).

11.2.11. Let \( r > 0 \), \( (a, b) \in \mathbb{R}^2 \), \( f : B_r(a, b) \to \mathbb{R} \), and suppose that the first-order partial derivatives \( f_x \) and \( f_y \) exist in \( B_r(a, b) \) and are differentiable at \( (a, b) \).

a) Set \( \Delta(h) = f(a + h, b + h) - f(a + h, b) - f(a, b + h) + f(a, b) \) and prove for \( h \) sufficiently small that

\[
\frac{\Delta(h)}{h} = f_y(a + h, b + th) - f_y(a, b) - \nabla f_y(a, b) \cdot (h, th) \\
- (f_x(a, b + th) - f_x(a, b) - \nabla f_x(a, b) \cdot (0, th)) + hf_{yx}(a, b)
\]

for some \( t \in (0, 1) \).

b) Prove that

\[
\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b).
\]

c) Prove that

\[
\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).
\]

11.3 DERIVATIVES, DIFFERENTIALS, AND TANGENT PLANES

In this section we begin to explore the analogy between \( Df \) and \( f' \). First we examine how the total derivative interacts with the algebra of functions.

11.20 Theorem. Let \( \alpha \in \mathbb{R} \), \( \mathbf{a} \in \mathbb{R}^n \), and suppose that \( f \) and \( g \) are vector functions. If \( f \) and \( g \) are differentiable at \( \mathbf{a} \), then \( f + g, \alpha f, \) and \( f \cdot g \) are all differentiable at \( \mathbf{a} \). In fact,

\[
D(f + g)(\mathbf{a}) = Df(\mathbf{a}) + Dg(\mathbf{a}), \tag{7}
\]
\[
D(\alpha f)(\mathbf{a}) = \alpha Df(\mathbf{a}), \tag{8}
\]

and

\[
D(f \cdot g)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a}). \tag{9}
\]

[The sums which appear on the right side of (7) and (9) represent matrix addition, and the products which appear on the right side of (9) represent matrix multiplication.]