Proof. If $(x, y) \neq (0, 0)$, then we can use the one-dimensional Product Rule to verify that both f_x and f_y exist and are continuous, for example,

$$f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \cos \frac{1}{\sqrt{x^2 + y^2}} + 2x \sin \frac{1}{\sqrt{x^2 + y^2}}.$$

Thus f is differentiable on $\mathbf{R}^2 \setminus \{(0, 0)\}$. Since $f_x(x, 0)$ has no limit as $x \to 0$, the partial derivative f_x is not continuous at $(0, 0)$. A similar statement holds for f_y. Thus to check differentiability at $(0, 0)$ we must return to the definition.

First, we compute the partial derivatives at $(0, 0)$. By definition,

$$f_x(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \to 0} t \sin \frac{1}{|t|} = 0,$$

and similarly, $f_y(0, 0) = 0$. Thus, both first partials exist at $(0, 0)$ and $\nabla f(0, 0) = (0, 0)$.

To prove that f is differentiable at $(0, 0)$, we must verify (4) for $\mathbf{a} = (0, 0)$. But it is clear that

$$\frac{f(h, k) - f(0, 0) - \nabla f(0, 0) \cdot (h, k)}{\|(h, k)\|} = \sqrt{h^2 + k^2} \sin \frac{1}{\sqrt{h^2 + k^2}} \to 0$$

as $(h, k) \to (0, 0)$. Thus f is differentiable at $(0, 0)$. ■

EXERCISES

11.2.1. Suppose, for $j = 1, 2, \dots, n$, that f_j are real functions continuously differentiable on the interval $(-1, 1)$. Prove that

$$g(\mathbf{x}) := f_1(x_1) \cdots f_n(x_n)$$

is differentiable on the cube $(-1, 1) \times (-1, 1) \times \cdots \times (-1, 1)$.

11.2.2. Suppose that $\mathbf{f}, \mathbf{g} : \mathbf{R} \to \mathbf{R}^m$ are differentiable at a and there is a $\delta > 0$ such that $\mathbf{g}(x) \neq \mathbf{0}$ for all $0 < |x - a| < \delta$. If $\mathbf{f}(a) = \mathbf{g}(a) = \mathbf{0}$ and $D\mathbf{g}(a) \neq \mathbf{0}$, prove that

$$\lim_{x \to a} \frac{\|\mathbf{f}(x)\|}{\|\mathbf{g}(x)\|} = \frac{\|D\mathbf{f}(a)\|}{\|D\mathbf{g}(a)\|}.$$

11.2.3. Prove that $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$.

11.2.4. Prove that

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{\sin \sqrt{x^2 + y^2}} & 0 < \|(x, y)\| < \pi \\ 0 & (x, y) = (0, 0) \end{cases}$$

is not differentiable at $(0, 0)$.

11.2.5. Prove that

$$f(x,y) = \begin{cases} \dfrac{x^4+y^4}{(x^2+y^2)^\alpha} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable on $\mathbf{R}^2$ for all $\alpha < 3/2$.

11.2.6. Prove that if $\alpha > 1/2$, then

$$f(x,y) = \begin{cases} |xy|^\alpha \log(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$.

11.2.7. Prove that

$$f(x,y) = \begin{cases} \dfrac{x^3-xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbf{R}^2$ and has first-order partial derivatives everywhere on $\mathbf{R}^2$, but f is not differentiable at $(0,0)$.

11.2.8. **This exercise is used several times in this chapter and the next.** Suppose that $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$. Prove that $\mathbf{T}$ is differentiable everywhere on $\mathbf{R}^n$ with

$$D\mathbf{T}(\mathbf{a}) = \mathbf{T} \quad \text{for} \quad \mathbf{a} \in \mathbf{R}^n.$$

11.2.9. Let $r > 0$, $f : B_r(\mathbf{0}) \to \mathbf{R}$, and suppose that there exists an $\alpha > 1$ such that $|f(\mathbf{x})| \leq \|\mathbf{x}\|^\alpha$ for all $\mathbf{x} \in B_r(\mathbf{0})$. Prove that f is differentiable at $\mathbf{0}$. What happens to this result when $\alpha = 1$?

11.2.10. Let V be open in $\mathbf{R}^n$, $\mathbf{a} \in V$, and $\mathbf{f} : V \to \mathbf{R}^m$.

***11.19 Definition.**

If $\mathbf{u}$ is a *unit* vector in $\mathbf{R}^n$ (i.e., $\|\mathbf{u}\| = 1$), then the *directional derivative* of $\mathbf{f}$ at $\mathbf{a}$ in the direction $\mathbf{u}$ is defined by

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{a}) := \lim_{t \to 0} \frac{\mathbf{f}(\mathbf{a}+t\mathbf{u}) - \mathbf{f}(\mathbf{a})}{t}$$

when this limit exists.

a) Prove that $D_{\mathbf{u}}\mathbf{f}(\mathbf{a})$ exists for $\mathbf{u} = \mathbf{e}_k$ if and only if $\mathbf{f}_{x_k}(\mathbf{a})$ exists, in which case

$$D_{\mathbf{e}_k}\mathbf{f}(\mathbf{a}) = \frac{\partial \mathbf{f}}{\partial x_k}(\mathbf{a}).$$

b) Show that if $\mathbf{f}$ has directional derivatives at $\mathbf{a}$ in all directions $\mathbf{u}$, then the first-order partial derivatives of $\mathbf{f}$ exist at $\mathbf{a}$. Use Example 11.11 to show that the converse of this statement is false.

11.3

c) Prove that the directional derivatives of

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

exist at $(0,0)$ in all directions $\mathbf{u}$, but f is neither continuous nor differentiable at $(0,0)$.

11.2.11. Let $r > 0$, $(a,b) \in \mathbf{R}^2$, $f : B_r(a,b) \to \mathbf{R}$, and suppose that the first-order partial derivatives f_x and f_y exist in $B_r(a,b)$ and are differentiable at (a,b).

a) Set $\Delta(h) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a,b)$ and prove for h sufficiently small that

$$\frac{\Delta(h)}{h} = f_y(a+h, b+th) - f_y(a,b) - \nabla f_y(a,b) \cdot (h, th)$$
$$- \big(f_y(a, b+th) - f_y(a,b) - \nabla f_y(a,b) \cdot (0, th)\big) + h f_{yx}(a,b)$$

for some $t \in (0,1)$.

b) Prove that

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a,b).$$

c) Prove that

$$\frac{\partial^2 f}{\partial x\, \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b).$$

11.3 DERIVATIVES, DIFFERENTIALS, AND TANGENT PLANES

In this section we begin to explore the analogy between $D\mathbf{f}$ and f'. First we examine how the total derivative interacts with the algebra of functions.

11.20 Theorem. *Let $\alpha \in \mathbf{R}$, $\mathbf{a} \in \mathbf{R}^n$, and suppose that $\mathbf{f}$ and $\mathbf{g}$ are vector functions. If $\mathbf{f}$ and $\mathbf{g}$ are differentiable at $\mathbf{a}$, then $\mathbf{f}+\mathbf{g}$, $\alpha\mathbf{f}$, and $\mathbf{f}\cdot\mathbf{g}$ are all differentiable at $\mathbf{a}$. In fact,*

$$D(\mathbf{f}+\mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{a}) + D\mathbf{g}(\mathbf{a}), \tag{7}$$
$$D(\alpha\mathbf{f})(\mathbf{a}) = \alpha D\mathbf{f}(\mathbf{a}), \tag{8}$$

and

$$D(\mathbf{f}\cdot\mathbf{g})(\mathbf{a}) = \mathbf{g}(\mathbf{a}) D\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{a}) D\mathbf{g}(\mathbf{a}). \tag{9}$$

[The sums which appear on the right side of (7) and (9) represent matrix addition, and the products which appear on the right side of (9) represent matrix multiplication.]