Math 125B, Winter 2015.

Homework 7 Solutions

11.2.2. Fix an $\epsilon > 0$. There exists an $\delta > 0$ so that

$$||f(x) - (x-a)Df(a)|| < \epsilon|x-a|$$

and

$$||g(x) - (x-a)Dg(a)|| < \epsilon|x-a|$$

provided that $|x-a| < \delta$. Thus

$$|x-a|\,||Df(a)|| - \epsilon|x-a| \le ||f(x)|| \le |x-a|\,||Df(a)|| + \epsilon|x-a|$$

and

$$|x-a|\,||Dg(a)|| - \epsilon|x-a| \le ||g(x)|| \le |x-a|\,||Dg(a)|| + \epsilon|x-a|.$$

Therefore

$$\frac{||Df(a)|| - \epsilon}{||Dg(a)|| + \epsilon} \le \frac{||f(x)||}{||g(x)||} \le \frac{||Df(a)|| + \epsilon}{||Dg(a)|| - \epsilon}.$$

It follows that

$$\limsup_{x\to a} \frac{||f(x)||}{||g(x)||} \le \frac{||Df(a)|| + \epsilon}{||Dg(a)|| - \epsilon}$$

and, as $\epsilon > 0$ is arbitrary,

$$\limsup_{x\to a} \frac{||f(x)||}{||g(x)||} \le \frac{||Df(a)||}{||Dg(a)||}.$$

Similarly

$$\liminf_{x\to a} \frac{||f(x)||}{||g(x)||} \ge \frac{||Df(a)||}{||Dg(a)||}.$$

Thus the lim sup and lim inf are equal, and their common value is the limit.

11.2.3. We have that $f(x,0) = 0$ for every x; thus $f_x(0,0) = 0$, and similarly (or by symmetry) $f_y(0,0) = 0$. If f is to be differentiable, $Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$, and so

$$0 = \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}.$$

If $(x,y) = r(a,b)$, with $a^2 + b^2 = 1$, then the above expression is $|ab|$ and so the limit does not exist. Therefore f is not differentiable at $(0,0)$.

11.2.4. We have that $f(x,0) = x^2/\sin|x|$ for every $x \neq 0$ and of course $f(0,0) = 0$. Call $g(x) = f(x,0)$ and let's try to compute $g'(0) = f_x(0,0)$. By definition, $g'(0)$ equals the limit, as $x \to 0$, of

$$\frac{g(x)}{x} = \frac{x}{\sin|x|}.$$

However, the limit of this expression as $x \to 0+$ is 1 and limit as $x \to 0-$ is -1. Therefore the limit as $x \to 0$ does not exist, $f_x(0,0)$ does not exist and thus $Df(0,0)$ does not exist.

11.2.5. Clearly, $f \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$, so the only issue is differentiability at $(0,0)$. For $x \neq 0$, $f(x,0) = |x|^{4-2\alpha}$. As $x \to 0$, $|f(x,0)/x| = |x|^{3-2\alpha} \to 0$ and so $f_x(0,0) = 0$. By symmetry $f_y(0,0) = 0$. Thus we need to show that

$$0 = \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{(x^2+y^2)^{1/2+\alpha}}.$$

If $(x,y) = r(a,b)$, with $a^2+b^2 = 1$, then the above expression is positive and equals $r^{3-2\alpha}(a^4+b^4) \leq 2r^{3-2\alpha}$ which is independent of a and b and goes to 0 as $r \to 0$ (as $3 - 2\alpha > 0$). Therefore f is differentiable at $(0,0)$ with derivative $Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

Remark 1. Another (a bit less general) method to solve this problem is to show that $f \in \mathcal{C}^1(\mathbb{R}^2)$. Compute

$$f_x(x,y) = \frac{2x(2x^2y^2 - \alpha x^4 - \alpha y^4 + 2x^4)}{(x^2+y^2)^{\alpha+1}}$$

for $(x,y) \neq (0,0)$ and, as we have seen that $f_x(0,0) = 0$, we need to show that

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = 0.$$

By symmetry, this will hold for f_y as well. Using the same method as above, we get $|f_x(x,y)| = r^{3-2\alpha}|2a(2a^2b^2 - \alpha a^4 - \alpha b^4 + 2a^4)| \leq 14\, r^{3-2\alpha}$, which is independent of a and b and goes to 0 as $r \to 0$. Therefore $f \in \mathcal{C}^1(\mathbb{R}^2)$ and consequently differentiable.

Remark 2. The function is not differentiable for $\alpha \geq 3/2$. The easiest way to see this is to look at the directional derivative

$$D_{(a,b)}f(0,0) = \lim_{t\to 0} t^{3-2\alpha} \frac{a^4+b^4}{(a^2+b^2)^\alpha},$$

which does not exist if $\alpha > 3/2$; when $\alpha = 3/2$, it exists but is clearly not a linear function in (a,b).

11.2.7. As $f \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$, the only issue is continuity and differentiability at $(0,0)$. If we write $(x,y) = r(a,b)$, with $a^2+b^2 = 1$, then $f(x,y) = r(a^3 - ab^2)$ and so $|f(x,y)| \leq 2r$, and so f is continuous at $(0,0)$. As $f(x,0) = x$ for every x, $f_x(0,0) = 1$, and as $f(0,y) = 0$ for every y, $f_y(0,0) = 0$. If f is differentiable at $(0,0)$, then

$$0 = \lim_{(x,y)\to(0,0)} \frac{f(x,y) - x}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{-2xy^2}{(x^2+y^2)^{3/2}}.$$

But the above limit does not exist, as if again $(x,y) = r(a,b)$ with $a^2+b^2 = 1$, the expression is $-2ab^2$.

Remark. Another way to prove non-differentiability is through directional derivative

$$D_{(a,b)}f(0,0) = \lim_{t\to 0} \frac{f(ta,tb)}{t} = a^3 - ab^2,$$

which is not a linear function of (a,b).