11.2.2. Fix an \( \epsilon > 0 \). There exists an \( \delta > 0 \) so that

\[
||f(x) - (x - a)Df(a)|| < \epsilon|x - a|
\]

and

\[
||g(x) - (x - a)Dg(a)|| < \epsilon|x - a|
\]

provided that \( |x - a| < \delta \). Thus

\[
|x - a||Df(a)| - \epsilon|x - a| \leq ||f(x)|| \leq |x - a||Df(a)| + \epsilon|x - a|
\]

and

\[
|x - a||Dg(a)| - \epsilon|x - a| \leq ||g(x)|| \leq |x - a||Dg(a)| + \epsilon|x - a|.
\]

Therefore

\[
\frac{||Df(a)|| - \epsilon}{||Dg(a)|| + \epsilon} \leq \frac{||f(x)||}{||g(x)||} \leq \frac{||Df(a)|| + \epsilon}{||Dg(a)|| - \epsilon}.
\]

It follows that

\[
\limsup_{x \to a} \frac{||f(x)||}{||g(x)||} \leq \frac{||Df(a)|| + \epsilon}{||Dg(a)|| - \epsilon}
\]

and, as \( \epsilon > 0 \) is arbitrary,

\[
\limsup_{x \to a} \frac{||f(x)||}{||g(x)||} \leq \frac{||Df(a)||}{||Dg(a)||}.
\]

Similarly

\[
\liminf_{x \to a} \frac{||f(x)||}{||g(x)||} \geq \frac{||Df(a)||}{||Dg(a)||}.
\]

Thus the \( \limsup \) and \( \liminf \) are equal, and their common value is the limit.

11.2.3. We have that \( f(x, 0) = 0 \) for every \( x \); thus \( f_x(0, 0) = 0 \), and similarly (or by symmetry) \( f_y(0, 0) = 0 \). If \( f \) is to be differentiable, \( Df(0, 0) = [0 \ 0] \), and so

\[
0 = \lim_{(x,y) \to (0,0)} \frac{f(x,y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{\sqrt{|xy|}}{\sqrt{x^2 + y^2}}
\]

If \((x,y) = r(a,b), \) with \( a^2 + b^2 = 1, \) then the above expression is \( |ab| \) and so the limit does not exist. Therefore \( f \) is not differentiable at \((0,0)\).

11.2.4. We have that \( f(x, 0) = x^2/ \sin |x| \) for every \( x \neq 0 \) and of course \( f(0, 0) = 0 \). Call \( g(x) = f(x, 0) \) and let’s try to compute \( g'(0) = f_x(0, 0) \). By definition, \( g'(0) \) equals the limit, as \( x \to 0, \) of

\[
\frac{g(x)}{x} = \frac{x}{\sin |x|}.
\]
However, the limit of this expression as \( x \to 0^+ \) is 1 and limit as \( x \to 0^- \) is \(-1\). Therefore the limit as \( x \to 0 \) does not exist, \( f_x(0, 0) \) does not exist and thus \( Df(0, 0) \) does not exist.

11.2.5. Clearly, \( f \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\}) \), so the only issue is differentiability at \((0, 0)\). For \( x \neq 0 \), \( f(x, 0) = |x|^{1-2\alpha} \). As \( x \to 0 \), \( |f(x, 0)/x| = |x|^{3-2\alpha} \to 0 \) and so \( f_x(0, 0) = 0 \). By symmetry \( f_y(0, 0) = 0 \). Thus we need to show that

\[
0 = \lim_{(x,y) \to (0,0)} \frac{f(x,y)}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{x^4 + y^4}{(x^2 + y^2)^{1/2+\alpha}}.
\]

If \((x, y) = (a, b)\), with \(a^2 + b^2 = 1\), then the above expression is positive and equals \(r^{3-2\alpha}(a^4 + b^4) \leq 2r^{3-2\alpha}\) which is independent of \(a\) and \(b\) and goes to 0 as \(r \to 0\) (as \(3 - 2\alpha > 0\)). Therefore \(f\) is differentiable at \((0, 0)\) with derivative \(Df(0, 0) = [0 \ 0]\).

**Remark 1.** Another (a bit less general) method to solve this problem is to show that \( f \in C^1(\mathbb{R}^2) \). Compute

\[
f_x(x, y) = \frac{2x(2x^2y^2 - \alpha x^4 - \alpha y^4 + 2x^4)}{(x^2 + y^2)^{\alpha+1}}
\]

for \((x, y) \neq (0, 0)\) and, as we have seen that \(f_x(0, 0) = 0\), we need to show that

\[
\lim_{(x,y) \to (0,0)} f_x(x,y) = 0.
\]

By symmetry, this will hold for \(f_y\) as well. Using the same method as above, we get \(|f_x(x, y)| = r^{3-2\alpha}|2a(2a^2b^2 - \alpha a^4 - \alpha b^4 + 2a^4)| \leq 14r^{3-2\alpha}\), which is independent of \(a\) and \(b\) and goes to 0 as \(r \to 0\). Therefore \(f \in C^1(\mathbb{R}^2)\) and consequently differentiable.

**Remark 2.** The function is not differentiable for \(\alpha \geq 3/2\). The easiest way to see this is to look at the directional derivative

\[
D_{(a,b)}f(0,0) = \lim_{t \to 0} |t^{3-2\alpha}| \frac{a^4 + b^4}{(a^2 + b^2)^\alpha},
\]

which does not exist if \(\alpha > 3/2\); when \(\alpha = 3/2\), it exists but is clearly not a linear function in \((a, b)\).

11.2.7. As \(f \in C^\infty(\mathbb{R}^2 \setminus \{(0, 0)\})\), the only issue is continuity and differentiability at \((0, 0)\). If we write \((x, y) = (a, b)\), with \(a^2 + b^2 = 1\), then \(f(x, y) = r(a^3 - ab^2)\) and so \(|f(x, y)| \leq 2r\), and so \(f\) is continuous at \((0, 0)\). As \(f(x, 0) = x\) for every \(x\), \(f_x(0, 0) = 1\), and as \(f(0, y) = 0\) for every \(y\), \(f_y(0, 0) = 0\). If \(f\) is differentiable at \((0, 0)\), then

\[
0 = \lim_{(x,y) \to (0,0)} \frac{f(x,y) - x}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \frac{-2xy^2}{(x^2 + y^2)^{3/2}}.
\]

But the above limit does not exist, as if again \((x, y) = (a, b)\) with \(a^2 + b^2 = 1\), the expression is \(-2ab^2\).

**Remark.** Another way to prove non-differentiability is through directional derivative

\[
D_{(a,b)}f(0,0) = \lim_{t \to 0} \frac{f(ta, tb)}{t} = a^3 - ab^2,
\]

which is not a linear function of \((a, b)\).