The Chain Rule can be used to compute individual partial derivatives without writing out the entire matrices $D\mathbf{f}$ and $D\mathbf{g}$. For example, suppose that $f(u_1, \ldots, u_m)$ is differentiable from $\mathbf{R}^m$ to $\mathbf{R}$, that $\mathbf{g}(x_1, \ldots, x_n)$ is differentiable from $\mathbf{R}^n$ to $\mathbf{R}^m$, and that $z = f(\mathbf{g}(x_1, \ldots, x_n))$. Since $Df = \nabla f$ and the jth column of $D\mathbf{g}$ consists of first partial derivatives, with respect to x_j, of the components $u_k := g_k(x_1, \ldots, x_n)$, it follows from the Chain Rule and the definition of matrix multiplication that

$$\frac{\partial z}{\partial x_j} = \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \cdots + \frac{\partial f}{\partial u_m}\frac{\partial u_m}{\partial x_j}$$

for $j = 1, 2, \ldots, n$. Here are two concrete examples which illustrate this principle.

11.29 *EXAMPLES.*

i) If $F, G, H : \mathbf{R}^2 \to \mathbf{R}$ are differentiable and $z = F(x, y)$, where $x = G(r, \theta)$, and $y = H(r, \theta)$, then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}.$$

ii) If $f : \mathbf{R}^3 \to \mathbf{R}$ and $\phi, \psi, \sigma : \mathbf{R} \to \mathbf{R}$ are differentiable and $w = f(x, y, z)$, where $x = \phi(t)$, $y = \psi(t)$, and $z = \sigma(t)$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

EXERCISES

11.4.1. Let $F : \mathbf{R}^3 \to \mathbf{R}$ and $f, g, h : \mathbf{R}^2 \to \mathbf{R}$ be $\mathcal{C}^2$ functions. If $w = F(x, y, z)$, where $x = f(p, q)$, $y = g(p, q)$, and $z = h(p, q)$, find formulas for w_p, w_q, and w_{pp}.

11.4.2. Let $r > 0$, let $\mathbf{a} \in \mathbf{R}^n$, and suppose that $\mathbf{g} : B_r(\mathbf{a}) \to \mathbf{R}^m$ is differentiable at $\mathbf{a}$.

a) If $f : B_r(g(\mathbf{a})) \to \mathbf{R}$ is differentiable at $\mathbf{g}(\mathbf{a})$, prove that the partial derivatives of $h = f \circ \mathbf{g}$ are given by

$$\frac{\partial h}{\partial x_j}(\mathbf{a}) = \nabla f(\mathbf{g}(\mathbf{a})) \cdot \frac{\partial \mathbf{g}}{\partial x_j}(\mathbf{a})$$

for $j = 1, 2, \ldots, n$.

b) If $n = m$ and $\mathbf{f} : B_r(g(\mathbf{a})) \to \mathbf{R}^n$ is differentiable at $\mathbf{g}(\mathbf{a})$, prove that

$$\det(D(\mathbf{f} \circ \mathbf{g})(\mathbf{a})) = \det(D\mathbf{f}(\mathbf{g}(\mathbf{a})))\det(D\mathbf{g}(\mathbf{a})).$$

11.4.3. Suppose that $k \in \mathbf{N}$ and that $f : \mathbf{R}^n \to \mathbf{R}$ is homogeneous of order k; that is, that $f(\rho\mathbf{x}) = \rho^k f(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^n$ and all $\rho \in \mathbf{R}$. If f is differentiable on $\mathbf{R}^n$, prove that

$$x_1 \frac{\partial f}{\partial x_1}(\mathbf{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\mathbf{x}) = kf(\mathbf{x})$$

for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$.

11.4.4. Let $f, g : \mathbf{R} \to \mathbf{R}$ be twice differentiable. Prove that $u(x, y) := f(xy)$ satisfies

$$x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = 0,$$

and $v(x, y) := f(x - y) + g(x + y)$ satisfies the *wave equation*; that is,

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0.$$

11.4.5. Let $f, g : \mathbf{R}^2 \to \mathbf{R}$ be differentiable and satisfy the *Cauchy–Riemann equations*; that is, that

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$

hold on $\mathbf{R}^2$. If $u(r, \theta) = f(r\cos\theta, r\sin\theta)$ and $v(r, \theta) = g(r\cos\theta, r\sin\theta)$, prove that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad r \neq 0.$$

11.4.6. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be $\mathcal{C}^2$ on $\mathbf{R}^2$ and set $u(r, \theta) = f(r\cos\theta, r\sin\theta)$. If f satisfies the *Laplace equation*; that is, if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

prove for each $r \neq 0$ that

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = 0.$$

11.4.7. Let

$$u(x, t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}, \qquad t > 0,\ x \in \mathbf{R}.$$

a) Prove that u satisfies the *heat equation* (i.e., $u_{xx} - u_t = 0$ for all $t > 0$ and $x \in \mathbf{R}$).

b) If $a > 0$, prove that $u(x,t) \to 0$, as $t \to 0+$, uniformly for $x \in [a, \infty)$.

11.4.8. Let $u : \mathbf{R} \to [0, \infty)$ be differentiable. Prove that for each $(x, y, z) \neq (0, 0, 0)$,

$$F(x, y, z) := u\left(\sqrt{x^2 + y^2 + z^2}\right)$$

satisfies

$$\left(\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right)^{1/2} = \left|u'\left(\sqrt{x^2 + y^2 + z^2}\right)\right|.$$

11.4.9. Suppose that $z = F(x, y)$ is differentiable at (a, b), that $F_y(a, b) \neq 0$, and that I is an open interval containing a. Prove that if $f : I \to \mathbf{R}$ is differentiable at a, $f(a) = b$, and $F(x, f(x)) = 0$ for all $x \in I$, then

$$\frac{df}{dx}(a) = \frac{-\dfrac{\partial F}{\partial x}(a, b)}{\dfrac{\partial F}{\partial y}(a, b)}.$$

11.4.10. Suppose that I is a nonempty, open interval and that $\mathbf{f} : I \to \mathbf{R}^m$ is differentiable on I. If $\mathbf{f}(I) \subseteq \partial B_r(\mathbf{0})$ for some fixed $r > 0$, prove that $\mathbf{f}(t)$ is orthogonal to $\mathbf{f}'(t)$ for all $t \in I$.

11.4.11. Let V be open in $\mathbf{R}^n$, $\mathbf{a} \in V$, $f : V \to \mathbf{R}$, and suppose that f is differentiable at $\mathbf{a}$.

a) Prove that the directional derivative $D_{\mathbf{u}} f(\mathbf{a})$ exists (see Exercise 11.2.10) for each $\mathbf{u} \in \mathbf{R}^n$ such that $\|\mathbf{u}\| = 1$ and $D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$.

b) If $\nabla f(\mathbf{a}) \neq \mathbf{0}$ and θ represents the angle between $\mathbf{u}$ and $\nabla f(\mathbf{a})$, prove that $D_{\mathbf{u}} f(\mathbf{a}) = \|\nabla f(\mathbf{a})\| \cos\theta$.

c) Show that as $\mathbf{u}$ ranges over all unit vectors in $\mathbf{R}^n$, the maximum of $D_{\mathbf{u}} f(\mathbf{a})$ is $\|\nabla f(\mathbf{a})\|$, and it occurs when $\mathbf{u}$ is parallel to $\nabla f(\mathbf{a})$.

11.5 THE MEAN VALUE THEOREM AND TAYLOR'S FORMULA

Using $D\mathbf{f}$ as a replacement for f', we guess that the multidimensional analogue of the Mean Value Theorem is $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) = D\mathbf{f}(\mathbf{c})(\mathbf{x} - \mathbf{a})$ for some $\mathbf{c}$ "between" $\mathbf{x}$ and $\mathbf{a}$; that is, some $\mathbf{c} \in L(\mathbf{x}; \mathbf{a})$, the line segment from $\mathbf{a}$ to $\mathbf{x}$. The following result shows that this guess is correct when $\mathbf{f}$ is real valued (see also Exercises 11.5.6 and 11.5.9).

11.30 Theorem. [MEAN VALUE THEOREM FOR REAL VALUED FUNCTIONS].
Let V be open in $\mathbf{R}^n$ and suppose that $f : V \to \mathbf{R}$ is differentiable on V. If $\mathbf{x}, \mathbf{a} \in V$ and $L(\mathbf{x}; \mathbf{a}) \subset V$, then there is a $\mathbf{c} \in L(\mathbf{x}; \mathbf{a})$ such that

$$f(\mathbf{x}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{a}). \tag{24}$$