$$f(\mathbf{x})-f(\mathbf{a})=\sum_{k=1}^{p-1}\frac{1}{k!}D^{(k)}f(\mathbf{a};\mathbf{h})+\frac{1}{(p-1)!}\int_0^1(1-t)^{p-1}D^{(p)}f(\mathbf{a}+t\mathbf{h};\mathbf{h})\,dt.$$

11.5.6. Let $r > 0$, $a, b \in \mathbf{R}$, $f : B_r(a, b) \to \mathbf{R}$ be differentiable, and $(x, y) \in B_r(a, b)$.

a) Let $g(t) = f(tx + (1-t)a, y) + f(a, ty + (1-t)b)$ and compute the derivative of g.

b) Prove that there are numbers c between a and x, and d between b and y such that

$$f(x, y) - f(a, b) = (x - a)f_x(c, y) + (y - b)f_y(a, d).$$

(This is Exercise 12.20 in Apostol [1].)

11.5.7. Suppose that $0 < r < 1$ and that $f : B_1(\mathbf{0}) \to \mathbf{R}$ is continuously differentiable. If there is an $\alpha > 0$ such that $|f(\mathbf{x})| \le \|\mathbf{x}\|^\alpha$ for all $\mathbf{x} \in B_r(\mathbf{0})$, prove that there is an $M > 0$ such that $|f(\mathbf{x})| \le M\|\mathbf{x}\|$ for $\mathbf{x} \in B_r(\mathbf{0})$.

11.5.8. Suppose that V is open in $\mathbf{R}^n$, that $f : V \to \mathbf{R}$ is $\mathcal{C}^2$ on V, and that $f_{x_j}(\mathbf{a}) = 0$ for some $\mathbf{a} \in H$ and all $j = 1, \dots, n$. Prove that if H is a compact convex subset of V, then there is a constant M such that

$$|f(\mathbf{x}) - f(\mathbf{a})| \le M\|\mathbf{x} - \mathbf{a}\|^2$$

for all $\mathbf{x} \in H$.

11.5.9. Let $f : \mathbf{R}^n \to \mathbf{R}$. Suppose that for each unit vector $\mathbf{u} \in \mathbf{R}^n$, the directional derivative $D_{\mathbf{u}}f(\mathbf{a} + t\mathbf{u})$ exists for $t \in [0, 1]$ (see Definition 11.19). Prove that

$$f(\mathbf{a} + \mathbf{u}) - f(\mathbf{a}) = D_{\mathbf{u}}f(\mathbf{a} + t\mathbf{u})$$

for some $t \in (0, 1)$.

11.5.10. Suppose that V is open in $\mathbf{R}^2$, that $(a, b) \in V$, and that $f : V \to \mathbf{R}$ is $\mathcal{C}^3$ on V. Prove that

$$\lim_{r\to 0}\frac{4}{\pi r^2}\int_0^{2\pi} f(a+r\cos\theta, b+r\sin\theta)\cos(2\theta)\,d\theta = f_{xx}(a, b) - f_{yy}(a, b).$$

11.5.11. Suppose that V is open in $\mathbf{R}^2$, that $H = [a, b] \times [0, c] \subset V$, that $u : V \to \mathbf{R}$ is $\mathcal{C}^2$ on V, and that $u(x_0, t_0) \ge 0$ for all $(x_0, t_0) \in \partial H$.

a) Show that, given $\varepsilon > 0$, there is a compact set $K \subset H^o$ such that $u(x, t) \ge -\varepsilon$ for all $(x, t) \in H \setminus K$.

b) Suppose that $u(x_1, t_1) = -\ell < 0$ for some $(x_1, t_1) \in H^o$, and choose $r > 0$ so small that $2rt_1 < \ell$. Apply part a) to $\varepsilon := \ell/2 - rt_1$ to choose the compact set K, and prove that the minimum of

$$w(x, t) := u(x, t) + r(t - t_1)$$

on H occurs at some $(x_2, t_2) \in K$.