Section 11.5  The Mean Value Theorem and Taylor’s Formula 423

\[ f(x) - f(a) = \sum_{k=1}^{p-1} \frac{1}{k!} D^k f(a; h) + \frac{1}{(p-1)!} \int_0^1 (1-t)^{p-1} D^p f(a+th; h) \, dt. \]

11.5.6. Let \( r > 0 \), \( a, b \in \mathbb{R} \), \( f : B_r(a, b) \to \mathbb{R} \) be differentiable, and \((x, y) \in B_r(a, b)\).

a) Let \( g(t) = f(tx + (1-t)a, y) + f(a, ty + (1-t)b) \) and compute the derivative of \( g \).

b) Prove that there are numbers \( c \) between \( a \) and \( x \), and \( d \) between \( b \) and \( y \) such that

\[ f(x, y) - f(a, b) = (x - a) f_x(c, y) + (y - b) f_y(a, d). \]

(This is Exercise 12.20 in Apostol [1].)

11.5.7. Suppose that \( 0 < r < 1 \) and that \( f : B_1(0) \to \mathbb{R} \) is continuously differentiable. If there is an \( M > 0 \) such that \( |f(x)| \leq \|x\|^a \) for all \( x \in B_r(0) \), prove that there is a \( M > 0 \) such that \( |f(x)| \leq M \|x\| \) for all \( x \in B_r(0) \).

11.5.8. Suppose that \( V \) is open in \( \mathbb{R}^n \), that \( f : V \to \mathbb{R} \) is \( C^2 \) on \( V \), and that \( f_{xy}(a) = 0 \) for some \( a \in H \) and all \( j = 1, \ldots, n \). Prove that if \( H \) is a compact convex subset of \( V \), then there is a constant \( M \) such that

\[ |f(x) - f(a)| \leq M \|x - a\|^2 \]

for all \( x \in H \).

11.5.9. Let \( f : \mathbb{R}^n \to \mathbb{R} \). Suppose that for each unit vector \( u \in \mathbb{R}^n \), the directional derivative \( D_u f(a + tu) \) exists for \( t \in [0, 1] \) (see Definition 11.19). Prove that

\[ f(a + tu) - f(a) = D_u f(a + tu) \]

for some \( t \in (0, 1) \).

11.5.10. Suppose that \( V \) is open in \( \mathbb{R}^2 \), that \( (a, b) \in V \), and that \( f : V \to \mathbb{R} \) is \( C^3 \) on \( V \). Prove that

\[ \lim_{r \to 0} \frac{1}{\pi r^2} \int_0^{2\pi} f(a + r \cos \theta, b + r \sin \theta) \cos(2\theta) \, d\theta = f_{xx}(a, b) - f_{xy}(a, b). \]

11.5.11. Suppose that \( V \) is open in \( \mathbb{R}^2 \), that \( H = [a, b] \times [0, c] \subset V \), that \( u : V \to \mathbb{R} \) is \( C^2 \) on \( V \), and that \( u(x_0, t_0) \geq 0 \) for all \( (x_0, t_0) \in \partial H \).

a) Show that, given \( \varepsilon > 0 \), there is a compact set \( K \subset H^o \) such that \( u(x, t) \geq -\varepsilon \) for all \( (x, t) \in H \setminus K \).

b) Suppose that \( u(x_1, t_1) = -\ell < 0 \) for some \((x_1, t_1) \in H^o \), and choose \( r > 0 \) so small that \( 2rt_1 < \ell \). Apply part a) to \( \varepsilon := \ell / 2 - rt_1 \) to choose the compact set \( K \), and prove that the minimum of

\[ w(x, t) := u(x, t) + r(t - t_1) \]

on \( H \) occurs at some \((x_2, t_2) \in K \).