Then $\mathbf{F}(4, 3, 2, 1, -1, -2) = (0, 0)$, and

$$\frac{\partial(F_1, F_2)}{\partial(u, v)} = \det\begin{bmatrix} 2u & 2v \\ 2u/x^2 & 2v/y^2 \end{bmatrix} = 4uv\left(\frac{1}{y^2} - \frac{1}{x^2}\right).$$

This determinant is nonzero when $u = 4$, $v = 3$, $x = 2$, and $y = 1$. Therefore, such functions u, v exist by the Implicit Function Theorem. ■

EXERCISES

11.6.1. For each of the following functions, prove that $\mathbf{f}^{-1}$ exists and is differentiable in some nonempty, open set containing (a, b), and compute $D(\mathbf{f}^{-1})(a, b)$

a) $\mathbf{f}(u, v) = (3u - v, 2u + 5v)$ at any $(a, b) \in \mathbf{R}^2$
b) $\mathbf{f}(u, v) = (u + v, \sin u + \cos v)$ at $(a, b) = (0, 1)$
c) $\mathbf{f}(u, v) = (uv, u^2 + v^2)$ at $(a, b) = (2, 5)$
d) $\mathbf{f}(u, v) = (u^3 - v^2, \sin u - \log v)$ at $(a, b) = (-1, 0)$

11.6.2. For each of the following functions, find out whether the given expression can be solved for z in a nonempty, open set V containing $(0, 0, 0)$. Is the solution differentiable near $(0, 0)$?

a) $xyz + \sin(x + y + z) = 0$
b) $x^2 + y^2 + z^2 + \sqrt{\sin(x^2 + y^2) + 3z + 4} = 2$
c) $xyz(2\cos y - \cos z) + (z\cos x - x\cos y) = 0$
d) $x + y + z + g(x, y, z) = 0$, where g is any continuously differentiable function which satisfies $g(0, 0, 0) = 0$ and $g_z(0, 0, 0) > 0$

11.6.3. Prove that there exist functions $u(x, y)$, $v(x, y)$, and $w(x, y)$, and an $r > 0$ such that u,v,w are continuously differentiable and satisfy the equations

$$u^5 + xv^2 - y + w = 0$$
$$v^5 + yu^2 - x + w = 0$$
$$w^4 + y^5 - x^4 = 1$$

on $B_r(1, 1)$, and $u(1, 1) = 1$, $v(1, 1) = 1$, $w(1, 1) = -1$.

11.6.4. Find conditions on a point (x_0, y_0, u_0, v_0) such that there exist real-valued functions $u(x, y)$ and $v(x, y)$ which are continuously differentiable near (x_0, y_0) and satisfy the simultaneous equations

$$xu^2 + yv^2 + xy = 9$$
$$xv^2 + yu^2 - xy = 7.$$

Prove that the solutions satisfy $u^2 + v^2 = 16/(x + y)$.