Math 125B, Winter 2015.

**Notes on inverse functions**

**Theorem 1** (Inverse Function Theorem)**.** *Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function in $\mathcal{C}^r(A)$ for some $r \geq 1$. Assume that $a \in A$ and that $\det Df(a) \neq 0$. Then there exist open sets $U \subseteq A$ and $V \subseteq \mathbb{R}^n$ so that $a \in U$, $f : U \to V$ is one-to-one and onto, and $f^{-1} : V \to U$ is in $\mathcal{C}^r(V)$. Moreover, if $y \in V$, and $x \in U$ with $f(x) = y$, $Df(x)$ is invertible and*

$$Df^{-1}(y) = Df(x)^{-1}. \tag{1}$$

*Remark.* The exponent $-1$ on the right of (1) is the inverse of the $n \times n$ matrix $Df(x)$, while on the left of (1) it indicates the inverse function.

*Definition.* Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function. We say that $f$ has a *differentiable local inverse* at $a \in A$ if there exist open sets $U \subseteq A$ and $V \subseteq \mathbb{R}^n$ so that $a \in U$, $f : U \to V$ is one-to-one and onto, and $f^{-1} : V \to U$ is differentiable on $V$.

The Inverse Function Theorem gives the sufficient condition for existence of differentiable local inverse. You should note that $f^{-1}$ may exist without being differentiable. For example, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is in $\mathcal{C}^\infty(\mathbb{R})$ and has inverse $f^{-1}(x) = x^{1/3}$ that is not differentiable at $x = 0$. The next lemma implies (1) assuming the rest of the Inverse Function Theorem holds.

**Lemma 1.** *Assume $A, B \subseteq \mathbb{R}^n$ is open and $f : A \to B$ is one-to-one and onto. Assume that $a \in A$, $b = f(a)$, and that $Df(a)$ and $Df^{-1}(b)$ both exist. Then $Df(a)$ is invertible and*

$$Df^{-1}(b) = Df(a)^{-1}.$$

*Proof.* If we denote by $I : \mathbb{R}^n \to \mathbb{R}^n$ the (linear) identity map, then on $A$

$$f^{-1} \circ f = I,$$

and then by the chain rule

$$Df^{-1}(b) \cdot Df(a) = I,$$

which ends the proof. $\square$

Lemma 1 and Theorem 1 together give the following result, which is the one you should remember for the exams.

**Theorem 2.** *Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function in $\mathcal{C}^1(A)$. Then $f$ has a differentiable local inverse at $a \in A$ if and only if $\det Df(a) \neq 0$ and in this case $Df^{-1}(f(a)) = Df(a)^{-1}$.*

We now proceed through the proof of Theorem 1, which does require some effort.

**Lemma 2.** *Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function in $\mathcal{C}^1(A)$ for some $r \geq 1$. Assume that $a \in A$ and that* $\det Df(a) \neq 0$. *Then there exists an open set $U \subseteq A$ with $a \in U$ and an $\alpha > 0$ so that*

$$||f(x_1) - f(x_2)|| \geq \alpha||x_1 - x_2|| \tag{2}$$

*for all $x_1, x_2 \in U$. In particular $f$ is one-to-one on $U$.*

*Proof.* Let $T = Df(a)$. Then

$$||x_1 - x_2|| = ||T^{-1}(Tx_1 - Tx_2)|| \leq M||Tx_1 - Tx_2||$$

for some $M > 0$. Pick $\alpha = 1/(2M)$.

Now let $g(x) = f(x) - Tx$. Then, on $A$, $Dg = Df - T$ and so $Dg(a) = 0$. As $g \in \mathcal{C}^1(A)$, we can choose an $\epsilon > 0$ so that the absolute value of all entries of $Dg$ is strictly less than $\alpha/n^2$ for $x \in B_\epsilon(a)$. This ball will be our open set $U$.

Fix an $i \in \{1, \ldots, n\}$. By the Mean Value Theorem, for any $x_1, x_2 \in U$, there exists a $c \in U$ so that[1]

$$|g_i(x_1) - g_i(x_2)| = |\nabla g_i(c) \cdot (x_1 - x_2)| \leq n \cdot \frac{\alpha}{n^2}||x_1 - x_2|| = \frac{\alpha}{n}||x_1 - x_2||.$$

Therefore, $||g(x_1) - g(x_2)|| \leq \alpha||x_1 - x_2||$, and then

$$\begin{aligned}
\alpha||x_1 - x_2|| &\geq ||g(x_1) - g(x_2)|| \\
&= ||f(x_1) - Tx_1 - f(x_2) + Tx_2|| \\
&\geq ||Tx_1 - Tx_2|| - ||f(x_1) - f(x_2)|| \\
&\geq 2\alpha||x_1 - x_2|| - ||f(x_1) - f(x_2)||,
\end{aligned}$$

from which the claimed inequality follows. □

The most difficult part of the proof of Inverse Function Theorem is the next preliminary result.

**Lemma 3.** *Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function in $\mathcal{C}^r(A)$ for some $r \geq 1$. Let $B = f(A)$. If $f$ is one-to-one on $A$, and* $\det Df(x) \neq 0$ *for all $x \in A$, then $B$ is open and $f^{-1} : B \to A$ is in $\mathcal{C}^r(B)$.*

*Proof.* We divide the proof into several steps.

*Step 1*: $B$ is open.

Pick a $b \in B$, and let $a = f^{-1}(B) \in A$. Then pick an $r > 0$ so that $Q = B_r(a) \subseteq A$. Then $\partial Q$ is compact and the so is $f(\partial Q)$, which is therefore closed. Moreover, as $f$ is one-to-one, $b \notin f(\partial Q)$. Thus there is a $\delta > 0$ so that $B_{2\delta}(b) \cap f(\partial Q) = \emptyset$.

Pick an arbitrary $y \in B_\delta(b)$ we will show that there is an $x_0 \in A$ so that $f(x_0) = y$, which will show that $B_\delta(b) \subseteq f(A)$ and establish Step 1. Let $\varphi(x) = ||f(x) - y||^2$. Then $\varphi : A \to \mathbb{R}$ is in $\mathcal{C}^r(A)$. As $Q$ is compact, $\varphi$ achieves minimum on $Q$, say at $x_0 \in Q$. Note that

$$\varphi(a) = ||f(a) - y||^2 = ||b - y||^2 < \delta^2,$$

and so $\min_Q \varphi < \delta^2$. On the other hand, if $x \in \partial Q$, then

$$||f(x) - y|| \geq ||f(x) - b|| - ||b - y|| \geq 2\delta - \delta = \delta,$$

[1] Recall that for $x \in \mathbb{R}^n$, $||x||_\infty = \max |x_i|$ and $||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$.

which implies that $x_0 \notin \partial Q$, and thus is a local minimum in the interior of $Q$. This implies[2] that $f(x_0) = y$.

*Step 2*: $g = f^{-1}$ is continuous.

We need to show that for every open set $U \subseteq A$, $g^{-1}(U)$ is open. But $g^{-1}(U) = f(U)$, which is open by Step 1.

*Step 3*: $g = f^{-1}$ is differentiable on $B$.

Let $T = Df(a)$. We need to show that

$$\frac{g(b+k) - g(b) - T^{-1}k}{||k||} \to 0 \tag{3}$$

as $k \to 0$.

By Lemma 2, there is an open set $U$ containing $a$, and some $\alpha > 0$, such that

$$||f(x_2) - f(x_1)|| \geq \alpha ||x_2 - x_1||$$

for all $x_1, x_2 \in U$. By Step 1, there exists an $\epsilon > 0$ so that $b + k \in f(U)$ whenever $||k|| < \epsilon$. Thus, when $||k|| < \epsilon$,

$$||k|| = ||b + k - b|| \geq \alpha ||g(b+k) - g(b)||$$

which we rewrite

$$\frac{||g(b+k) - g(b)||}{||k||} \leq \frac{1}{\alpha}. \tag{4}$$

We rewrite the expression in (3)

$$\frac{g(b+k) - g(b) - T^{-1}k}{||k||} = -T^{-1}\left(\frac{k - T(g(b+k) - g(b))}{||g(b+k) - g(b)||}\right), \cdot \frac{||g(b+k) - g(b)||}{||k||}$$

and we use (4); also, because $-T^{-1}$ is linear we obtain an $M > 0$ so that

$$\frac{||g(b+k) - g(b) - T^{-1}k||}{||k||} \leq \frac{M}{\alpha} \cdot \frac{||k - T(g(b+k) - g(b))||}{||g(b+k) - g(b)||}.$$

Now we let $h = g(b+k) - g(b) = g(b+k) - a$. By continuity of $g$ (Step 2), $h \to 0$ as $k \to 0$. Observe that $b + k = f(a+h)$ and so $k = f(a+h) - f(a)$. Therefore,

$$\frac{||g(b+k) - g(b) - T^{-1}k||}{||k||} \leq \frac{M}{\alpha} \cdot \frac{||f(a+h) - f(a) - Th||}{||h||} \to 0 \tag{5}$$

as $h \to 0$, because $T = Df(a)$. Clearly, (5) implies (3).

*Step 4*: $g = f^{-1} \in \mathcal{C}^r(B)$.

We first assume that $r = 1$ and show that $g \in \mathcal{C}^1(B)$. We know that $Dg(x) = Df(g(x))^{-1}$. Moreover, the entries of the inverse of an invertible matrix $T \in \mathbb{R}^{n,n}$ are rational, thus $\mathcal{C}^\infty$, functions of entries of $T$. As $g$ is continuous, and the entries of $Df$ are continuous, this implies that the entries of $Dg$ are continuous.

Now we proceed by induction on $r$. Assume the claim holds for functions in $\mathcal{C}^{r-1}$. If $f$ is in $\mathcal{C}^r$, then certainly $f \in \mathcal{C}^{r-1}$, and $g \in \mathcal{C}^{r-1}$ by the induction hypothesis. Also, $Df$ is in $\mathcal{C}^{r-1}$ (or more precisely its entries are) and so $Dg$, as a composite function of two $\mathcal{C}^{r-1}$ functions and a $\mathcal{C}^\infty$ function (the inverse) is also in $\mathcal{C}^{r-1}$. Thus $g$ is in $\mathcal{C}^r$. □

[2]See Problem 7 in Discussion Problems 9.

*Proof of Theorem 1.* By Lemma 2, there is an open set $U_1 \subseteq A$, with $a \in U_1$, on which $f$ is one-to-one. As $\det Df(x)$ is a continuous function of $x$ on $A$ (as a polynomial in the entries of $Df(x)$), there is a open set $U_2$, with $a \in U_2$, so that $\det Df(x) \neq 0$ for $x \in U_2$. Take $U = U_1 \cap U_2$. Lemma 3 implies that this set has the propertied claimed in Theorem 1. Finally, as already remarked, (1) follows by Lemma 1. $\square$

Next is the most famous consequence of the Inverse Function Theorem. In it, we view an $\mathbb{R}^m$-valued function $f$ defined on a subset of $\mathbb{R}^{n+m}$ as a function on $\mathbb{R}^n \times \mathbb{R}^m$. That is, $f$ is given by $f(x, y)$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. The next theorem tells us when can we the relation $f(x, y) = 0$ defines a differentiable function $y = g(x)$ from $n$ variables to $m$ variables. We will denote $D_y f$ to be the derivative of $f$ with respect to $y$ when $x$ is held constant, that is, the $m \times m$ matrix of partial derivatives of $f$ with respect to coordinates of $y$. Similarly, $D_x f(x, y)$ is the $m \times n$ matrix of partial derivatives of $f$ with respect to coordinates of $x$. We should note that we assume that the coordinates are arranged so that the last $m$ of them are those of $y$. That is, of course not necessary: the coordinates of $y$ could be the first $m$, or any other among the $n + m$ inputs of $f$.

**Theorem 3** (Implicit Function Theorem)**.** *Assume $G \subset \mathbb{R}^{n+m}$ is open and $f : G \to \mathbb{R}^m$ is in $\mathcal{C}^r(G)$ for some $r \geq 1$. Suppose $(a, b) \in G$ is such that $f(a, b) = 0$ and $\det D_y f(a, b) \neq 0$. Then there is an open set $A \subset \mathbb{R}^n$ containing $a$, an open set $B \subset \mathbb{R}^m$ containing $b$, and a function $g : A \to B$ in $\mathcal{C}^r(A)$ so that $g(a) = b$ and $y = g(x)$ is the unique solution of the equation $f(x, y) = 0$ for all $x \in A$ and $y \in B$. Moreover, for $x \in A$ and $y \in B$, $\det D_y f(x, y) \neq 0$ and*

$$Dg(x) = D_y f(x, g(x))^{-1} D_x f(x, g(x)). \tag{6}$$

*Proof.* Define $F : G \to \mathbb{R}^{n+m}$ by $F(x, y) = (x, f(x, y))$. Then $F \in \mathcal{C}^1(G)$ and, for $(x, y) \in G$, $\det DF(x, y) = \det D_y f(x, y)$. Therefore $DF(a, b) \neq 0$. The Implicit Function Theorem implies that there exists an open set $U$ containing $(a, b)$, on which $F$ has a differentiable inverse $H$; $H$ maps an open set containing $F(a, b) = (a, 0)$ into $U$. We can find open sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, so that $a \in A$, $b \in B$, and $A \times B \subset U$. By continuity, we can also assume that $\det D_y f(x, y) \neq 0$ on $U$. Given the form of $F$, $H$ must be of the form $H(x, y) = (x, h(x, y))$ for some function $h$ in $\mathcal{C}^r$. Define the (linear) projection function $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^m$ by $\pi(x, y) = y$. Then $f = \pi \circ F$ and

$$f(x, h(x, y)) = (f \circ H)(x, y) = ((\pi \circ F) \circ H)(x, y) = (\pi \circ (F \circ H))(x, y) = \pi(x, y) = y.$$

It follows that, if we define $g(x) = h(x, 0)$, we get $f(x, g(x)) = 0$. The formula (6) for the derivative follows by the chain rule, by differentiating the equation $f(x, g(x)) = 0$. $\square$