Notes on inverse functions

Theorem 1 (Inverse Function Theorem). Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function in $C^r(A)$ for some $r \geq 1$. Assume that $a \in A$ and that $\det Df(a) \neq 0$. Then there exist open sets $U \subseteq A$ and $V \subseteq \mathbb{R}^n$ so that $a \in U$, $f : U \to V$ is one-to-one and onto, and $f^{-1} : V \to U$ is in $C^r(V)$. Moreover, if $y \in V$, and $x \in U$ with $f(x) = y$, $Df(x)$ is invertible and

\[ Df^{-1}(y) = Df(x)^{-1}. \]

Remark. The exponent $-1$ on the right of (1) is the inverse of the $n \times n$ matrix $Df(x)$, while on the left of (1) it indicates the inverse function.

Definition. Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function. We say that $f$ has a differentiable local inverse at $a \in A$ if there exist open sets $U \subseteq A$ and $V \subseteq \mathbb{R}^n$ so that $a \in U$, $f : U \to V$ is one-to-one and onto, and $f^{-1} : V \to U$ is differentiable on $V$.

The Inverse Function Theorem gives the sufficient condition for existence of differentiable local inverse. You should note that $f^{-1}$ may exist without being differentiable. For example, $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is in $C^\infty(\mathbb{R})$ and has inverse $f^{-1}(x) = x^{1/3}$ that is not differentiable at $x = 0$. The next lemma implies (1) assuming the rest of the Inverse Function Theorem holds.

Lemma 1. Assume $A, B \subseteq \mathbb{R}^n$ is open and $f : A \to B$ is one-to-one and onto. Assume that $a \in A$, $b = f(a)$, and that $Df(a)$ and $Df^{-1}(b)$ both exist. Then $Df(a)$ is invertible and

\[ Df^{-1}(b) = Df(a)^{-1}. \]

Proof. If we denote by $I : \mathbb{R}^n \to \mathbb{R}^n$ the (linear) identity map, then on $A$

\[ f^{-1} \circ f = I, \]

and then by the chain rule

\[ Df^{-1}(b) \cdot Df(a) = I, \]

which ends the proof.

Lemma 1 and Theorem 1 together give the following result, which is the one you should remember for the exams.

Theorem 2. Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function in $C^1(A)$. Then $f$ has a differentiable local inverse at $a \in A$ if and only if $\det Df(a) \neq 0$ and in this case $Df^{-1}(f(a)) = Df(a)^{-1}$.

We now proceed through the proof of Theorem 1, which does require some effort.
Lemma 2. Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function in $C^1(A)$ for some $r \geq 1$. Assume that $a \in A$ and that $\det Df(a) \neq 0$. Then there exists an open set $U \subseteq A$ with $a \in U$ and an $\alpha > 0$ so that

$(2) \quad \|f(x_1) - f(x_2)\| \geq \alpha \|x_1 - x_2\|$

for all $x_1, x_2 \in U$. In particular $f$ is one-to-one on $U$.

Proof. Let $T = Df(a)$. Then

$$
||x_1 - x_2|| = ||T^{-1}(Tx_1 - Tx_2)|| \leq M||Tx_1 - Tx_2||
$$

for some $M > 0$. Pick $\alpha = 1/(2M)$.

Now let $g(x) = f(x) - Tx$. Then, on $A$, $Dg = Df - T$ and so $Dg(a) = 0$. As $g \in C^1(A)$, we can choose an $\epsilon > 0$ so that the absolute value of all entries of $Dg$ is strictly less than $\alpha/n^2$ for $x \in B_\epsilon(a)$. This ball will be our open set $U$.

Fix an $i \in \{1, \ldots, n\}$. By the Mean Value Theorem, for any $x_1, x_2 \in U$, there exists a $c \in U$ so that

$$
|g_i(x_1) - g_i(x_2)| = |\nabla g_i(c) \cdot (x_1 - x_2)| \leq n \cdot \frac{\alpha}{n^2} \|x_1 - x_2\| = \frac{\alpha}{n} \|x_1 - x_2\|.
$$

Therefore, $\|g(x_1) - g(x_2)\| \leq \alpha \|x_1 - x_2\|$, and then

$$
\alpha \|x_1 - x_2\| \geq \|g(x_1) - g(x_2)\|
$$

$$
= \|f(x_1) - Tx_1 - f(x_2) + Tx_2\|
$$

$$
\geq \|Tx_1 - Tx_2\| - \|f(x_1) - f(x_2)\|
$$

$$
\geq 2\alpha \|x_1 - x_2\| - \|f(x_1) - f(x_2)\|,
$$

from which the claimed inequality follows.

The most difficult part of the proof of Inverse Function Theorem is the next preliminary result.

Lemma 3. Assume $A \subseteq \mathbb{R}^n$ is open and $f : A \to \mathbb{R}^n$ is a function in $C^r(A)$ for some $r \geq 1$. Let $B = f(A)$. If $f$ is one-to-one on $A$, and $\det Df(x) \neq 0$ for all $x \in A$, then $B$ is open and $f^{-1} : B \to A$ is in $C^r(B)$.

Proof. We divide the proof into several steps.

Step 1: $B$ is open.

Pick a $b \in B$, and let $a = f^{-1}(B) \in A$. Then pick an $r > 0$ so that $Q = B_{\epsilon}(a) \subseteq A$. Then $\partial Q$ is compact and so is $f(\partial Q)$, which is therefore closed. Moreover, as $f$ is one-to-one, $b \notin f(\partial Q)$. Thus there is a $\delta > 0$ so that $B_{\delta}(b) \cap f(\partial Q) = \emptyset$.

Pick an arbitrary $y \in B_{\delta}(b)$ we will show that there is an $x_0 \in A$ so that $f(x_0) = y$, which will show that $B_{\delta}(b) \subseteq f(A)$ and establish Step 1. Let $\varphi(x) = \|f(x) - y\|^2$. Then $\varphi : A \to \mathbb{R}$ is in $C^r(A)$. As $Q$ is compact, $\varphi$ achieves minimum on $Q$, say at $x_0 \in Q$. Note that

$$
\varphi(a) = \|f(a) - y\|^2 = \|b - y\|^2 < \delta^2,
$$

and so $\min_Q \varphi < \delta^2$. On the other hand, if $x \in \partial Q$, then

$$
\|f(x) - y\| \geq \|f(x) - b\| - \|b - y\| \geq 2\delta - \delta = \delta,
$$

Recall that for $x \in \mathbb{R}^n$, $\|x\|_{\infty} = \max |x_i|$ and $\|x\|_{\infty} \leq \|x\|_2 \leq \sqrt{n} \|x\|_{\infty}$. 2
which implies that \( x_0 \notin \partial Q \), and thus is a local minimum in the interior of \( Q \). This implies\(^2\) that \( f(x_0) = y \).

\textbf{Step 2: }\( g = f^{-1} \) is continuous.

We need to show that for every open set \( U \subseteq A \), \( g^{-1}(U) \) is open. But \( g^{-1}(U) = f(U) \), which is open by Step 1.

\textbf{Step 3: }\( g = f^{-1} \) is differentiable on \( B \).

Let \( T = Df(a) \). We need to show that

\[
\frac{g(b + k) - g(b) - T^{-1}k}{||k||} \to 0
\]
as \( k \to 0 \).

By Lemma 2, there is an open set \( U \) containing \( a \), and some \( \alpha > 0 \), such that

\[
||f(x_2) - f(x_1)|| \geq \alpha ||x_2 - x_1||
\]
for all \( x_1, x_2 \in U \). By Step 1, there exists an \( \epsilon > 0 \) so that \( b + k \in f(U) \) whenever \( ||k|| < \epsilon \). Thus, when \( ||k|| < \epsilon \),

\[
||k|| = ||b + k - b|| \geq \alpha ||g(b + k) - g(b)||
\]
which we rewrite

\[
\frac{||g(b + k) - g(b)||}{||k||} \leq \frac{1}{\alpha}.
\]

We rewrite the expression in (3)

\[
\frac{g(b + k) - g(b) - T^{-1}k}{||k||} = -T^{-1} \left( \frac{k - T(g(b + k) - g(b))}{||g(b + k) - g(b)||} \right), \frac{||g(b + k) - g(b)||}{||k||}
\]
and we use (4); also, because \( -T^{-1} \) is linear we obtain an \( M > 0 \) so that

\[
\frac{||g(b + k) - g(b) - T^{-1}k||}{||k||} \leq \frac{M}{\alpha} \cdot \frac{||k - T(g(b + k) - g(b))||}{||g(b + k) - g(b)||}.
\]
Now we let \( h = g(b + k) - g(b) = g(b + k) - a \). By continuity of \( g \) (Step 2), \( h \to 0 \) as \( k \to 0 \). Observe that \( b + k = f(a + h) \) and so \( k = f(a + h) - f(a) \). Therefore,

\[
\frac{||g(b + k) - g(b) - T^{-1}k||}{||k||} \leq \frac{M}{\alpha} \cdot \frac{||f(a + h) - f(a) - Th||}{||h||} \to 0
\]
as \( h \to 0 \), because \( T = Df(a) \). Clearly, (5) implies (3).

\textbf{Step 4: }\( g = f^{-1} \in C^r(B) \).

We first assume that \( r = 1 \) and show that \( g \in C^1(B) \). We know that \( Dg(x) = Df(g(x))^{-1} \). Moreover, the entries of the inverse of an invertible matrix \( T \in \mathbb{R}^{n,n} \) are rational, thus \( C^\infty \), functions of entries of \( T \). As \( g \) is continuous, and the entries of \( Df \) are continuous, this implies that the entries of \( Dg \) are continuous.

Now we proceed by induction on \( r \). Assume the claim holds for functions in \( C^{r-1} \). If \( f \) is in \( C^r \), then certainly \( f \in C^{r-1} \), and \( g \in C^{r-1} \) by the induction hypothesis. Also, \( Df \) is in \( C^{r-1} \) (or more precisely its entries are) and so \( Dg \), as a composite function of two \( C^{r-1} \) functions and a \( C^\infty \) function (the inverse) is also in \( C^{r-1} \). Thus \( g \) is in \( C^r \).

\(^2\)See Problem 7 in Discussion Problems 9.
follows by the chain rule, by differentiating the equation $f$ and the form of $F$ open set containing $(x,y)$, there exists an open set $U$ such that $f$ is defined in $F$ and is such that $f(x,y) = 0$ for all $x \in A$ and $y \in B$. Moreover, for $x \in A$ and $y \in B$, $D_y f(x,y) \neq 0$ and

\begin{equation}
D_y f(x,y) = D_y f(x,g(x))^{-1} D_x f(x,g(x)).
\end{equation}

Proof. Define $F : G \to \mathbb{R}^{n+m}$ by $F(x,y) = (x,f(x,y))$. Then $F \in C^1(G)$ and, for $(x,y) \in G$, $D_F(x,y) = D_y f(x,y)$. Therefore $D_F(x,y) \neq 0$. The Implicit Function Theorem implies that there exists a function $g : A \to B$ in $C^r(A)$ such that $g(a) = b$ and $y = g(x)$ is the unique solution of the equation $f(x,y) = 0$ for all $x \in A$ and $y \in B$. Moreover, for $x \in A$ and $y \in B$, $D_y f(x,y) \neq 0$ and

\begin{equation}
D_y f(x,y) = D_y f(x,g(x))^{-1} D_x f(x,g(x)).
\end{equation}

Proof. Define $G : G \to \mathbb{R}^{n+m}$ by $G(x,y) = (f(x,y), h(x,y))$. Then $G \in C^1(G)$ and, for $(x,y) \in G$, $D_G(x,y) = D_x f(x,y)$ and $D_H(x,y) = D_y h(x,y)$. Therefore $D_G(x,y) \neq 0$. The Implicit Function Theorem implies that there exists a function $h : A \to B$ in $C^r(A)$ such that $h(a) = b$ and $y = h(x)$ is the unique solution of the equation $f(x,y) = 0$ for all $x \in A$ and $y \in B$. Moreover, for $x \in A$ and $y \in B$, $D_y h(x,y) \neq 0$ and

\begin{equation}
D_y f(x,y) = D_y f(x,g(x))^{-1} D_x f(x,g(x)).
\end{equation}

Next is the most famous consequence of the Inverse Function Theorem. In it, we view an $\mathbb{R}^m$-valued function $f$ defined on a subset of $\mathbb{R}^{n+m}$ as a function on $\mathbb{R}^n \times \mathbb{R}^m$. That is, $f$ is given by $f(x,y)$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. The next theorem tells us when can we the relation $f(x,y) = 0$ defines a differentiable function $y = g(x)$ from $n$ variables to $m$ variables. We will denote $D_y f$ to be the derivative of $f$ with respect to $y$ when $x$ is held constant, that is, the $m \times m$ matrix of partial derivatives of $f$ with respect to coordinates of $y$. Similarly, $D_x f(x,y)$ is the $n \times m$ matrix of partial derivatives of $f$ with respect to coordinates of $x$. We should note that we assume that the coordinates are arranged so that the last $m$ of them are those of $y$. That is, of course not necessary: the coordinates of $y$ could be the first $m$, or any other among the $n+m$ inputs of $f$.

Theorem 3 (Implicit Function Theorem). Assume $G \subset \mathbb{R}^{n+m}$ is open and $f : G \to \mathbb{R}^m$ is in $C^r(G)$ for some $r \geq 1$. Suppose $(a,b) \in G$ is such that $f(a,b) = 0$ and $D_y f(a,b) \neq 0$. Then there is an open set $A \subset \mathbb{R}^n$ containing $a$, an open set $B \subset \mathbb{R}^m$ containing $b$, and a function $g : A \to B$ in $C^r(A)$ so that $g(a) = b$ and $y = g(x)$ is the unique solution of the equation $f(x,y) = 0$ for all $x \in A$ and $y \in B$. Moreover, for $x \in A$ and $y \in B$, $D_y f(x,y) \neq 0$ and

\begin{equation}
D_g(x) = D_y f(x,g(x))^{-1} D_x f(x,g(x)).
\end{equation}

Proof. Define $H : H \to \mathbb{R}^m$ by $H(x,y) = (x,f(x,y))$. Then $H \in C^r(H)$ and, for $(x,y) \in H$, $D_H(x,y) = D_y f(x,y)$. Therefore $D_H(x,y) \neq 0$. The Implicit Function Theorem implies that there exists an open set $U$ containing $(a,b)$, on which $H$ is differentiable. Therefore $H$ maps an open set containing $H(a,b) = (a,0)$ into $U$. We can find open sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, so that $a \in A$, $b \in B$, and $A \times B \subset U$. By continuity, we can also assume that $D_y f(x,y) \neq 0$ on $U$. Given the form of $F$, $H$ must be of the form $H(x,y) = (x,h(x,y))$ for some function $h$ in $C^r$. Define the (linear) projection function $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^m$ by $\pi(x,y) = y$. Then $f = \pi \circ F$ and

\begin{equation}
f(x,h(x,y)) = (f \circ H)(x,y) = ((\pi \circ F) \circ H)(x,y) = (\pi \circ (F \circ H))(x,y) = \pi(x,y) = y.
\end{equation}

It follows that, if we define $g(x) = h(x,0)$, we get $f(x,g(x)) = 0$. The formula (6) for the derivative follows by the chain rule, by differentiating the equation $f(x,g(x)) = 0$. 

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