Math 125B, Winter 2015.

Multidimensional integration: definitions and main theorems

1 Definition

A (closed) rectangle $R \subseteq \mathbb{R}^n$ is a set of the form

$$R = [a_1, b_1] \times \dots \times [a_n, b_n] = \{(x_1, \dots, x_n) : a_1 \le x_1 \le b_1, \dots, a_n \le x_1 \le b_n\}.$$

Here $a_k \leq b_k$, for k = 1, ..., n. The (*n*-dimensional) volume of R is

$$\operatorname{Vol}(R) = \operatorname{Vol}_n(R) = (b_1 - a_1) \cdots (b_n - a_n).$$

We say that the rectangle is *nondegenerate* if Vol(R) > 0. A partition \mathcal{P} of R is induced by partition of each of the n intervals in each dimension. We identify the partition with the resulting set of closed subrectangles of R.

Assume $R \subseteq \mathbb{R}^n$ and $f : R \to \mathbb{R}$ is a bounded function. Then we define upper sum of f with respect to a partition \mathcal{P} , and upper integral of f on R,

$$U(f, \mathcal{P}) = \sum_{B \in \mathcal{P}} \sup_{B} f \cdot \operatorname{Vol}(B), \quad (U) \int_{R} f = \inf_{\mathcal{P}} U(f, \mathcal{P}),$$

and analogously lower sum of f with respect to a partition \mathcal{P} , and lower integral of f on R,

$$L(f, \mathcal{P}) = \sum_{B \in \mathcal{P}} \inf_{B} f \cdot \operatorname{Vol}(B), \quad (L) \int_{R} f = \sup_{\mathcal{P}} L(f, \mathcal{P}).$$

We call f integrable on R if $(U) \int_R f = (L) \int_R f$ and in this case denote the common value by $\int_R f = \int_R f(x) dx$. As in one-dimensional case, f is integrable on R if and only if Cauchy condition is satisfied. The Cauchy condition states that, for every $\epsilon > 0$, there exists a partition \mathcal{P} so that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon.$$

For $f : \mathbb{R}^n \to \mathbb{R}$, we define its *support*, $\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$. Then f has compact support if and only if it vanishes outside a bounded set. Assume that f is a bounded function with compact support. We define

$$\int_{\mathbb{R}^n} f = \int_R f$$

where R is any rectangle such that $\operatorname{supp} f \subseteq R$, provided the integral exists. The definition is justified by the next theorem.

Theorem 1 (Independence of Supporting Rectangle). If R and R' are two rectangles such that $\operatorname{supp} f \subseteq R$ and $\operatorname{supp} f \subseteq R'$ and one of the two integrals $\int_R f$ and $\int_{R'} f$ exists, then the other exists too, and both are equal.

Proof. Take any partition that refines $R \cap R'$. As $\operatorname{supp} f \subseteq R \cap R'$, the contribution of only subrectangle of $R \setminus (R \cap R')$ to the upper or to the lower sum is zero. Thus $\int_R f$ equals $\int_{R \cap R'} f$ and the same is true for $\int_{R'} f$.

If $f : \mathbb{R}^n \to \mathbb{R}$ has compact support and $\int_{\mathbb{R}^n}$ exists, we call f (*Riemann*) integrable on \mathbb{R}^n . We next define an integral of f over a subset of \mathbb{R}^n .

The *indicator* of a set $A \subseteq \mathbb{R}^n$ is the function $\chi_A : \mathbb{R}^n \to \mathbb{R}$ given by

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

If $f: A \to R$ is a function, we interpret $f\chi_A$ to be the function defined on \mathbb{R}^n by

$$(f\chi_A)(x) = \begin{cases} f(x) & x \in A\\ 0 & x \notin A \end{cases}$$

If A is bounded, and $f: A \to \mathbb{R}$ is a bounded function, we define

$$\int_A f = \int_{\mathbb{R}^n} f \chi_A$$

if it exists, that is, if $f\chi_A$ is integrable on \mathbb{R}^n . In this case, we say that f in *integrable on* A. We say that a bounded set A is *Jordan measurable* if $\int_{\mathbb{R}^n} \chi_A$ exist, in which case we define the *volume* of A to be

$$\operatorname{Vol}(A) = \operatorname{Vol}_n(A) = \int_{\mathbb{R}^n} \chi_A.$$

We note that integrable functions on \mathbb{R}^n satisfy the usual algebraic and monotonicity properties, inherited from the integral on a rectangle (and proved the same way as in the one-dimensional case). We will summarize some of them in the next theorem, but note immediately that if f is integrable on \mathbb{R}^n and $A \subseteq \mathbb{R}^n$ is Jordan measurable, then f is integrable on A, as $f\chi_A$ is a product of two Riemann integrable functions.

Theorem 2 (Linearity and Monotonicity). Assume $f, g : \mathbb{R}^n \to \mathbb{R}$ are bounded functions with compact support, Riemann integrable on \mathbb{R}^n . Assume also $A, B \subseteq \mathbb{R}^n$ are Jordan measurable.

- 1. For $a, b \in \mathbb{R}$, $\int_A (af + bg) = a \int_A f + b \int_A g$.
- 2. If $f \leq g$ on A, then $\int_A f \leq \int_A g$.
- 3. The function |f| is integrable on \mathbb{R}^n and $|\int_A f| \leq \int_A |f|$.
- 4. If $A \subseteq B$ and $f \ge 0$, then $\int_A f \le \int_B f$.
- 5. The sets $A \cap B$, $A \cup B$ and $A \setminus B$ are also Jordan measurable, and

$$\int_{A\cup B} f = \int_A f + \int_B f - \int_{A\cap B} f.$$

In particular, $\operatorname{Vol}(A \cup B) = \operatorname{Vol}(A) + \operatorname{Vol}(B) - \operatorname{Vol}(A \cap B)$.

Proof. We only prove the last assertion, others are also proved easily using an indicator argument. We observe that $\chi_{A\cap B} = \chi_A \cdot \chi_B$, $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$ and $\chi_{A\setminus B} = \chi_A - \chi_{A\cap B}$. To get the integral formula, integrate the equality $f\chi_{A\cup B} = f\chi_A + f\chi_B - f\chi_{A\cap B}$, and to get the volume formula, we apply the integral formula to $f = \chi_{A\cup B}$.

2 Existence

We say that a set $A \subseteq \mathbb{R}^n$ has measure zero (or is negligible) if for every $\epsilon > 0$ there exists countably many rectangles Q_1, Q_2, \ldots so that $A \subseteq \bigcup_k Q_k$ and $\sum_k \operatorname{Vol}(Q_k) < \epsilon$.

We get equivalent definitions if the rectangles are required to be open or closed. We will see below that a set or measure zero is not necessarily Jordan measurable, but if it is, its volume must be zero. The following famous theorem is the main result on Riemann integrability. We will not discuss its proof here.

Theorem 3 (Lebesgue Condition for Integrability). A bounded function f with compact support is Riemann integrable on \mathbb{R}^n if and only if the set of its discontinuities has measure zero.

We will now give some useful consequences of the above definition and theorem, with short summaries of their proofs.

Corollary 1. Every countable set has measure zero. More generally, if each of countably many sets A_1, A_2, \ldots has measure zero, their union $\cup_k A_k$ has measure zero.

Proof. Cover every set A_k with rectangles of combined length less than $\epsilon/2^k$, then gather all the coverings into a covering of the union, with combined length of size less than ϵ .

Example. If $S \subseteq \mathbb{R}^2$ is the set of points in $[0,1] \times [0,1]$ with rational coordinates, then S is not Jordan measurable, but is countable, so it has measure zero.

Theorem 4 (Measure Zero and Volume Zero). A nondegenerate rectangle does not have measure zero. More generally, any set of measure zero has empty interior. Moreover, if $S \subseteq \mathbb{R}^n$ is a Jordan measurable set of measure zero, and f is a Riemann integrable function on \mathbb{R}^n , then $\int_S f = 0$; consequently, $\operatorname{Vol}(S) = 0$. Conversely, if $S \subseteq \mathbb{R}^n$ is a Jordan measurable set of zero volume, then it has measure zero.

Proof. Fix an $\epsilon > 0$ such that $\epsilon < \operatorname{Vol}(R)$. Take a nondegenerate closed rectangle R, and assume that it has measure zero. Then cover it by open rectangles Q_k of combined volume less than ϵ . As R is compact, it is covered by finitely many of Q_k , say Q_1, \ldots, Q_n , which means that $\sum_{k=1}^n \chi_{Q_k} \ge 1$ on R. Then, by linearity of integral,

$$\operatorname{Vol}(R) = \int_{R} 1 \le \int_{R} \sum_{k=1}^{n} \chi_{Q_{k}} \le \sum_{k=1}^{n} \int_{R} \chi_{Q_{k}} \le \sum_{k=1}^{n} \operatorname{Vol}(Q_{k}) < \epsilon,$$

contradiction. It follows that a set of measure zero cannot contain a nondegenerate rectangle and thus has empty interior.

If S is Jordan measurable and f is integrable, then $f\chi_S$ is integrable. Take a rectangle R that encloses S, and a partition \mathcal{P} of R. Any $B \in \mathcal{P}$ must include a point outside S, and so $L(f\chi_S, \mathcal{P}) \leq 0$ and $U(f\chi_S, \mathcal{P}) \geq 0$. Therefore $(L) \int_R f\chi_S \leq 0$ and $(U) \int_R f\chi_S \geq 0$, and then, as $f\chi_S$ is integrable, $0 = \int_R f\chi_S = \int_S f$. In particular, $0 = \int_S \chi_S = \operatorname{Vol}(S)$.

To prove the last assertion, take any $\epsilon > 0$. Pick a rectangle R so that $S \subseteq R$. As $Vol(S) = \int_R \chi_S = 0$, there is a partition \mathcal{P} of R so that $U(\chi_S, \mathcal{P}) < \epsilon$. Observe that

$$U(\chi_S, \mathcal{P}) = \sum_{B \in \mathcal{P}: B \cap S \neq \emptyset} \operatorname{Vol}(B)$$

and so the cover of S that consists of rectangles $B \in \mathcal{P}$ that intersect S has combined volume less than ϵ .

Corollary 2. A bounded set $A \subseteq \mathbb{R}^n$ is Jordan measurable if and only if ∂A has measure 0. Consequently, if A is Jordan measurable, so are A° , \overline{A} and ∂A . Furthermore, if $f : \mathbb{R}^n \to \mathbb{R}$ is Riemann integrable on \mathbb{R}^n and $A \subseteq \mathbb{R}^n$ is Jordan measurable, then $\int_{\partial A} f = 0$, and $\int_A f = \int_{A^\circ} f = \int_{\overline{A}} f$

Proof. First claim follows from the fact that ∂A is exactly the set of discontinuities of χ_A . Jordan measurability of A° , \overline{A} , and ∂A follows because $\partial(A^\circ) \subseteq \partial A$, $\partial \overline{A} \subseteq \partial A$ and $\partial(\partial A) \subseteq \partial A$. As ∂A is Jordan measurable and has measure 0, $\int_{\partial A} f = 0$ by the previous theorem. So,

$$\int_{\overline{A}} f = \int_{A^{\circ}} f + \int_{\partial A} f = \int_{A^{\circ}} f.$$

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Moreover $A \cap \partial A$ is also Jordan measurable with measure 0 and then

$$\int_{A} f = \int_{A^{\circ}} f + \int_{A \cap \partial A} f = \int_{A^{\circ}} f.$$

Theorem 5 (Continuous Functions and Jordan Measurability). Assume $K \subseteq \mathbb{R}^n$ is compact and Jordan measurable. Assume also that $f, g: K \to \mathbb{R}^n$ are continuous and $g \leq f$ on K. Then

- (i) f is Riemann integrable on K;
- (ii) The region $L = \{(x, y) \in \mathbb{R}^{n+1} : x \in K \text{ and } g(x) \le y \le f(x)\}$ is Jordan measurable in \mathbb{R}^{n+1} .

Proof. The claim (i) follows from the Lebesgue's condition (Theorem 3). To prove (ii), we may assume that g = 0 and $f \ge 0$. Why? Assume we have proved the theorem for this case. We then observe that the theorem holds also for

$$L_1 = \{(x, y) \in \mathbb{R}^{n+1} : x \in K \text{ and } c \le y \le f(x)\},\$$

for arbitrary $c \in \mathbb{R}$, as we can reduce it to the case c = 0 by adding (0, -c). Then it also holds for

$$L_2 = \{(x, y) \in \mathbb{R}^{n+1} : x \in K \text{ and } c \le y \le g(x)\},\$$

and for

$$L_3 = \{(x, y) \in \mathbb{R}^{n+1} : x \in K \text{ and } c \le y < g(x)\}$$

because $\partial L_2 = \partial L_3$. Finally, of we choose $c = \inf_K g$, then $L = L_1 \setminus L_3$.

Let R be a rectangle with $K \subseteq R$ and \mathcal{P} a partition of R. (Define f(x) to be 0 for $x \notin K$.) Then

$$U(\chi_K f, \mathcal{P}) = \sum_{B \in \mathcal{P}} \sup_{B} (\chi_K f) \cdot \operatorname{Vol}_n(B) = \sum_{B \in \mathcal{P}: B \cap K \neq \emptyset} \sup_{B} f \cdot \operatorname{Vol}_n(B)$$

and

$$L(\chi_K f, \mathcal{P}) = \sum_{B \in \mathcal{P}: B \subseteq K} \inf_B f \cdot \operatorname{Vol}_n(B).$$

Let $\widetilde{R} = R \times [0, \sup_K f]$; this is a rectangle in \mathbb{R}^{n+1} and $L \subseteq \widetilde{R}$. In addition,

$$\bigcup_{B \in \mathcal{P}: B \subseteq K} B \times [0, \inf_K f] \subseteq L \subseteq \bigcup_{B \in \mathcal{P}: B \cap K \neq \emptyset} B \times [0, \sup_K f].$$

Let $\widetilde{\mathcal{P}}$ be a partition of \widetilde{R} which refines all rectangles $B \times [0, \inf_K f]$ for $B \subseteq K$, and all rectangles $B \times [0, \sup_K f]$ for $B \cap K \neq \emptyset$. Then

$$U(\chi_L, \widetilde{\mathcal{P}}) \le \sum_{B \in \mathcal{P}: B \cap K \neq \emptyset} \sup_B f \cdot \operatorname{Vol}_n(B) = U(\chi_K f, \mathcal{P})$$

and similarly

$$L(\chi_L, \mathcal{P}) \ge L(\chi_K f, \mathcal{P}).$$

So

$$U(\chi_L, \tilde{\mathcal{P}}) - L(\chi_L, \tilde{\mathcal{P}}) \le U(\chi_K f, \mathcal{P}) - L(\chi_K f, \mathcal{P})$$

By (i), for every $\epsilon > 0$ there exists a partition \mathcal{P} so that $U(\chi_K f, \mathcal{P}) - L(\chi_K f, \mathcal{P}) < \epsilon$, and so

$$U(\chi_L, \mathcal{P}) - L(\chi_L, \mathcal{P}) < \epsilon,$$

which proves (ii).

Corollary 3. In the setting of the previous theorem, let

$$graph(f) = \{(x, f(x)) : x \in K\} \subseteq \mathbb{R}^{n+1}$$

be the graph of f. Then graph(f) is Jordan measurable in \mathbb{R}^{n+1} and $\operatorname{Vol}_{n+1}(\operatorname{graph} f) = 0$.

Proof. Take first g to be a constant function, equal to any constant below $\inf_K f$ and consider the resulting L in the statement of the previous theorem. Then $\operatorname{graph}(f) \subseteq \partial L$, thus $\operatorname{graph}(f)$ has measure zero. Moreover, $\operatorname{graph}(f)$ is Jordan measurable, as we can see by taking g = f in the previous theorem.

3 Connection to Iterated Integrals

Theorem 6 (Fubini's Theorem). Assume $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are rectangles, and form the rectangle $R = A \times B \subseteq \mathbb{R}^{n+m}$. Assume that $f : R \to \mathbb{R}$ is bounded. We write f = f(x, y) for $x \in A$, $y \in B$. Define the functions $I_U, I_L : A \to \mathbb{R}$ by

$$I_U(x) = (U) \int_B f(x, y) \, dy, \quad I_L(x) = (L) \int_B f(x, y) \, dy.$$

Assume f is integrable on R. Then I_U and I_L are integrable on A and

$$\int_R f = \int_A I_U = \int_A I_L.$$

Proof. Let \mathcal{P} be a partition of R; then \mathcal{P} is given by

$$\mathcal{P} = \mathcal{P}_A \times \mathcal{P}_B = \{ R_A \times R_B : R_A \in \mathcal{P}_A, R_B \in \mathcal{P}_B \}$$

where \mathcal{P}_A is a partition of A and \mathcal{P}_B is a partition of B.

Step 1. $L(f, \mathcal{P}) \leq L(I_L, \mathcal{P}_A).$

The key is this obvious inequality

$$\inf_{R_A \times R_B} f \le \inf_{R_B} f(x_0, y), \text{ for every } x_0 \in R_A.$$

Therefore, for every $x_0 \in R_A$,

$$\sum_{R_B \in \mathcal{P}_B} \inf_{R_A \times R_B} f \cdot \operatorname{Vol}_m(R_B) \le L(f(x_0, \cdot), \mathcal{P}_B) \le (L) \int_B f(x_0, y) \, dy = I_L(x_0)$$

and so

$$\sum_{R_B \in \mathcal{P}_B} \inf_{R_A \times R_B} f \cdot \operatorname{Vol}_m(R_B) \le \inf_{R_A} I_L.$$

Multiply this by $\operatorname{Vol}_n(R_A)$ and sum over $R_A \in \mathcal{P}_A$ to get the claim in Step 1. Step 2. $U(f, \mathcal{P}) \geq U(I_L, \mathcal{P}_A)$.

This follows by the same argument as for Step 1.

Step 3. Conclusion of the proof.

It follows from Steps 1 and 2 that

$$(L) \int_{R} f \leq (L) \int_{A} I_{L} \leq (L) \int_{A} I_{U} \leq (U) \int_{A} I_{U} \leq (U) \int_{R} f$$
$$(L) \int_{R} f \leq (L) \int_{A} I_{L} \leq (U) \int_{A} I_{L} \leq (U) \int_{A} I_{U} \leq (U) \int_{R} f,$$

so all these are equal.

Corollary 4. In the setting of the previous theorem, if $\int_B f(x,y) dy$ exists (as a Riemann integral of the function $y \mapsto f(x,y)$) for every $x \in A$, then

$$\int_{R} f = \int_{A} \left(\int_{B} f(x, y) \, dy \right) \, dx.$$

Similarly, if $\int_A f(x, y) dx$ exists for every $y \in B$, then

$$\int_{R} f = \int_{B} \left(\int_{A} f(x, y) \, dx \right) \, dy.$$

Both of these are true when f is continuous on R.

Proof. The first two assertions follow directly from the Fubini's theorem. The last follows because continuity of f implies that $x \mapsto f(x, y)$ is continuous for every y, and $y \mapsto f(x, y)$ is continuous for every x.

4 Improper Integrals

The next example shows that a bounded continuous function on an open set is not necessarily integrable. This is one of the reasons improper integrals are needed; the statement of the change of variables theorem (see the next section) is very cumbersome if we are restricted to Jordan regions.

Example. Let q_1, q_2, \ldots be an enumeration of $\mathbb{Q} \cap (0, 1)$. Define G_k to be the open interval of length $1/2^{k+2}$ centered at q_k , that is

$$G_k = (q_k - 1/2^{k+3}, q_k + 1/2^{k+3}).$$

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Let

$$A = \bigcup_{k \ge 1} G_k \cap (0, 1).$$

Observe that the sum of the lengths of G_k is 1/4. Moreover, $A \subseteq (0, 1)$ is open as a union of open sets and $\overline{A} = [0, 1]$. Assume that ∂A has measure zero, which means that there are countably many open intervals of combined length at most 1/4 that cover ∂A . The open intervals in the definition of A, together with those that cover ∂A , cover $A \cup \partial A = \overline{A} = [0, 1]$, and have a combined length at most 1/4 + 1/4 = 1/2. As [0, 1] is compact, a finite selection I_1, \ldots, I_k of these covers [0, 1]. This means that $\chi_{I_1} + \ldots + \chi_{I_k} \geq \chi_{[0,1]}$, and now by taking the integral of both sides, we get $1/2 \geq \int \chi_{I_1} + \ldots + \int \chi_{I_k} \geq \int \chi_{[0,1]} = 1$, a contradiction. Thus ∂A does not have measure 0, A is not Jordan measurable and χ_A is not integrable. We will see that we can still define the length of A by approximation by compact Jordan measurable intervals from the inside. (Exact computation of the length depends on the enumeration of rational numbers in (0, 1) and is likely to be hard.)

Assume $G \subseteq \mathbb{R}^n$ is an open set, which is now *not* necessarily bounded. Assume $f : G \to [0, \infty)$. We say that f is *locally integrable* on G if it is integrable on K for every compact Jordan measurable $K \subseteq G$.¹ If f is continuous on G, it is locally integrable by Theorem 5. We do *not*, however, assume that f is bounded on G.

We will say that f has an *improper integral on* G if it is locally integrable and the following supremum over all compact Jordan measurable subsets $K \subseteq G$ is finite:

$$\sup_{K} \int_{K} f < \infty.$$

In this case we denote the supremum also as $\int_G f$. If $f: G \to \mathbb{R}$ is not necessarily positive, then we define

$$\int_G f = \int_G f_+ - \int_G f_-,$$

provided that the two integrals on the left side of the equation both exist. Thus the improper integral $\int_G f$ exists if and only if $\int_G |f| = \int_G f_+ + \int_G f_-$ exists. The improper integral shares a lot of properties of the proper one, and we now state the counterpart to Theorem 2,

Corollary 5. Assume $f, g : \mathbb{R}^n \to \mathbb{R}$ are functions whose improper integrals on open sets $A, B \subseteq \mathbb{R}^n$ exist.

- 1. For $a, b \in \mathbb{R}$, $\int_A (af + bg)$ exists and equals $a \int_A f + b \int_A g$.
- 2. If $f \leq g$ on A, then $\int_A f \leq \int_A g$.
- 3. The inequality $|\int_A f| \leq \int_A |f|$ holds.
- 4. If $A \subseteq B$ and $f \ge 0$, then $\int_A f \le \int_B f$.
- 5. The improper integrals $\int_{A \cap B} f$, and $\int_{A \cup B} f$ exist and

$$\int_{A\cup B} f = \int_A f + \int_B f - \int_{A\cap B} f.$$

Proof. All but last statement are routine exercises. The last one is also routine if $A \cap B = \emptyset$, but otherwise a little tricky, as one needs to find suitable compact subsets of $A \cap B$. As we will not really use it, we omit the proof.

¹By a standard compactness argument, f is locally integrable on G if and only every point $x \in G$ has is a center of a nondegenerate rectangle $R_x \subseteq G$ on which f is integrable. This is why we call this *local* integrability.

Theorem 7 (Bounded Functions on Bounded Open Sets). Assume $G \subseteq \mathbb{R}^n$ is bounded and open, and that $f: G \to \mathbb{R}$ is bounded and locally integrable on G. Then the improper integral $\int_G f$ exists. The condition holds when f is bounded and continuous on G.

Proof. If $|f| \leq M$, and G is included in a rectangle R, then the supremum in the definition is bounded by $M \cdot \text{Vol}(R)$.

As we use the same notation, it needs to be clear from the context whether the proper or the improper integral is meant. At least some of the confusion is cleared by the next theorem.

Theorem 8 (Proper and Improper Integrals). Assume $G \subseteq \mathbb{R}^n$ is open and bounded, and that a function $f: G \to \mathbb{R}$ is Riemann integrable on G (in the previous proper sense). Then the improper integral $\int_G f$ exists and equals to the proper integral $\int_G f$.

Proof. It is enough to prove this when $f \geq 0$. If $K \subseteq G$ is compact and Jordan measurable, then clearly $\int_K f \leq \int_G f$, so that $\sup_K \int_K f \leq \int_G f$. To show the opposite inequality, pick a rectangle Rthat includes G and a partition \mathcal{P} of R. Let $K_{\mathcal{P}}$ be the compact set obtained as the union of all rectangles B_1, \ldots, B_k of \mathcal{P} that are completely included in G. As the contribution to $L(f\chi_G, \mathcal{P})$ from all the other rectangles in \mathcal{P} is 0,

$$L(f\chi_G, \mathcal{P}) = \sum_{i=1}^k \inf_{B_i} f \cdot \operatorname{Vol}(B_i) \le \sum_{i=1}^k \int_{B_i} f = \int_{K_{\mathcal{P}}} f \le \sup_K \int_K f$$

and so

$$\int_{G} f = \sup_{\mathcal{P}} L(f\chi_{G}, \mathcal{P}) \le \sup_{K} \int_{K} f$$

and the two inequalities imply

$$\int_G f = \sup_K \int_K f.$$

In order to compute an improper integral, we need a sequence of sets that fill in G, and the next theorem provides a justification. The theorem is only of theoretical significance and we will not discuss it beyond its statement.

Theorem 9 (Improper Integrals and Nested Sequences I). Assume $G \subseteq \mathbb{R}^n$ is open, and $f : G \to [0,\infty)$ is locally integrable on G. Assume that we have a sequence of sets $K_i \subseteq G$ so that each K_i is compact and Jordan measurable, $K_i \subseteq K_{i+1}^\circ$ for each i, and $\cup_i K_i = G$. Define the nondecreasing sequence of numbers

$$I_i = \int_{K_i} f.$$

If the sequence I_i converges to a finite number, then the improper integral $\int_G f$ exists and equals the limit of this sequence. Conversely, if the sequence I_i diverges to ∞ , then the improper integral $\int_G f$ does not exist.

By far the most useful is the last theorem of this section, which the one to remember and use together with Theorem 8 and Corollary 2. Namely, the simplest way to compute an improper integral over an open region G is to write it as a suitable union of Jordan measurable open sets on which f is integrable in the proper sense.

Theorem 10 (Improper Integrals and Nested Sequences II). Assume $G \subseteq \mathbb{R}^n$ is open, and $f: G \to \mathbb{R}$ is continuous. Assume that $G_1 \subseteq G_2 \subseteq \cdots$ is a sequence of open sets whose union is G. Then the improper integral $\int_G f$ exists if and only each improper integral $I_k = \int_{G_k} |f|$ exists and the increasing sequence I_k converges. In this case,

$$\lim_{k \to \infty} \int_{G_k} f = \int_G f.$$

Proof. Again, it suffices to give a proof when $f \ge 0$. Assume first that $\int_G f < \infty$. Then $I_k = \int_{G_k} f \le \int_G f$ and so $I = \lim I_k$ exists and

$$I \le \int_G f.$$

Conversely, assume that $I = \lim I_k$ exists. Pick a compact Jordan measurable subset of G. Then K is covered by the sets G_k , hence by finitely many of them, hence (as they are nested) by one of them, say G_K . Then, by definition of the improper integral over the open set G_K ,

$$\int_{K} f \le \int_{G_K} f \le I$$

and therefore by definition of the improper integral over the open set G,

$$\int_G f \le I.$$

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5 Change of Variables

Assume $A, B \subseteq \mathbb{R}^n$ are open sets. We say that a function $g: A \to B$ is diffeomorphism if g is one-toone and onto, $g \in \mathcal{C}^1(A)$ and $g^{-1} \in \mathcal{C}^1(B)$. Remember that a necessary (but not sufficient) condition for g to be a diffeomorphism is that det $Dg(a) \neq 0$ for all $a \in A$.

The best formulation of our capstone theorem is in terms of open sets and thus in terms of improper integrals.

Theorem 11 (Change of Variables). Assume $A, B \subseteq \mathbb{R}^n$ are open sets and that a function $g : A \to B$ is a diffeomorphism. Assume $f : B \to \mathbb{R}^n$ is a function. Then f is integrable on B if and only if $(f \circ g) |\det Dg|$ is integrable on A and

$$\int_B f = \int_A (f \circ g) \cdot |\det Dg|.$$

All known proofs of this theorem are quite demanding and require a sophisticated localization argument. Therefore we merely explain why the theorem has this form. The aforementioned localization argument reduces the problem to the case where g is linear, in which case it is given by an invertible matrix T. Moreover, a further approximation argument implies reduction to the case where f is constant (which can then be taken to be 1 by linearity). The question then becomes: why is the volume of TA, for a Jordan measurable set A, equal to $|\det T|$ times the volume of A?

The best explanation (if not a completely rigorous proof) is through the very useful singular value decomposition of T. To review, this decomposition gives orthogonal matrices U and V and a diagonal matrix D, with strictly positive diagonal entries called singular values, so that

$$T = UDV^T$$
.

Since U and V^T are orthogonal matrices, their determinants are either 1 or -1. Therefore,

$$|\det T| = \det D.$$

Moreover, orthogonal transformations preserve the shape of sets and thus do not change the volume. The only transformation in the decomposition that changes the volume is D. How does, then, $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$ change the volume of a set A? By chopping A into small pieces, we may assume that A is a nondegenerate rectangle, say $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then $DA = [\lambda_1 a_1, \lambda_1 b_1] \times \cdots \times [\lambda_n a_n, \lambda_n b_n]$ and

$$\operatorname{Vol}(DA) = \prod_{k=1}^{n} (\lambda_k b_k - \lambda_k a_k) = \prod_{k=1}^{n} \lambda_k \cdot \prod_{k=1}^{n} (b_k - a_k) = \det D \cdot \operatorname{Vol}(A),$$

and so

 $\operatorname{Vol}(TA) = \det D \cdot \operatorname{Vol}(A) = |\det T| \cdot \operatorname{Vol}(A).$

This given a nice geometric meaning of the determinant: it measures how a linear map transforms the volume.