Math 125B, Winter 2015.
Feb. 4, 2015.

MIDTERM EXAM 1

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 5 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 6 pages (including this one) with 5 problems.

1	
2	
3	
4	
5	
TOTAL	

1. (a) Assume $f : [0,1] \to \mathbb{R}$ is a function and $P : 0 = x_0 < x_1 < \ldots < x_n = 1$ is a partition of $[0,1]$. State the definition of the lower Riemann sum $L(f,P)$ and upper Riemann sum $U(f,P)$. Then state precisely what this means: f is Riemann integrable on $[0,1]$.

$$U(f,P) = \sum_{j=1}^{n} \max_{[x_{j-1},x_j]} f \cdot (x_j - x_{j-1})$$

$$L(f,P) = \sum_{j=1}^{n} \inf_{[x_{j-1},x_j]} f \cdot (x_j - x_{j-1})$$

f is integrable $\iff$ ($\forall \varepsilon > 0$,) $\exists$ a partition P, so that $U(f,P) - L(f,P) < \varepsilon$

or $\iff \inf\limits_P U(f,P) = \sup\limits_P L(f,P)$, where the inf and sup are over all partitions P

(b) Compute

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{i^2+n^2}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i/n}{(i/n)^2+1} \cdot \frac{1}{n}$$

(as this is a Riemann sum for $f(x) = \frac{x}{x^2+1}$, and for partition of $[0,1]$ into n equal intervals, using the right endpts. as tags)

$$= \int_0^1 \frac{x}{1+x^2}\,dx \underset{\uparrow}{=} \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \underline{\underline{\frac{1}{2}\ln 2}}$$

FTC, as $\frac{d}{dx}\frac{1}{2}\ln(1+x^2) = \frac{x}{1+x^2}$

2. Assume $f : [0,1] \to \mathbb{R}$. For each statement below determine, with proof, whether it is true or false.
(a) If f is Riemann integrable on $[0,1]$, then it is continuous on $[0,1]$,.

No. For example, $f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0,1] \end{cases}$

is Riemann integrable, with Riemann integral 0.

(as it only has 1 discontinuity)

(c) If f is Riemann integrable on $[0,1]$, then so is the function g defined by $g(x) = \sin(f(x))$.

Yes. g is a composite of a Riemann integrable function f an continuous function sin, and is thus Riemann integrable.

3. Compute

$$\lim_{n\to\infty}\int_0^1 \underbrace{\frac{nx+x^3}{n+nx^2+x^4}}_{\substack{\shortparallel \\ f_n(x)}}\,dx$$

Let $f(x) = \dfrac{x}{1+x^2}$.

Then

$$\sup_{x\in[0,1]} \left| f_n(x) - f(x) \right|$$

$$= \sup_{x\in[0,1]} \left| \frac{nx+x^3}{n+nx^2+x^4} - \frac{x}{1+x^2} \right|$$

$$= \sup_{x\in[0,1]} \frac{\left| \cancel{nx} + x^3 + \cancel{nx^3} + \cancel{x^5} - \cancel{nx} - \cancel{nx^3} - \cancel{x^5} \right|}{(n+nx^2+x^4)(1+x^2)}$$

$$= \sup_{x\in[0,1]} \frac{x^3}{(n+nx^2+x^4)(1+x^2)}$$

$$\leq \frac{1}{n} \to 0$$

Thus $f_n \to f$ uniformly on $[0,1]$ and

$$\lim_{n\to\infty} \int f_n = \int f = \frac{1}{2}\ln 2 \quad (\text{by Problem 1b})$$

4. Assume that $f : \mathbb{R} \to \mathbb{R}$ has continuous derivative on $\mathbb{R}$, and that $f'(x) \geq 1$ for all $x \in [0, 1]$. Prove (using integration by parts or any other correct method) that

$$\int_0^1 x f(x)\, dx \leq \frac{1}{2} f(1) - \frac{1}{6}.$$

By parts: $u = f(x), \quad dv = x\,dx$

$du = f'(x)\,dx, \quad v = \frac{x^2}{2}$

$$\int_0^1 x f(x)\,dx = \frac{x^2}{2} f(x) \Big|_0^1 - \int_0^1 \frac{x^2}{2} \underbrace{f'(x)}_{\geq 1} dx$$ 10

$$\leq \frac{1}{2} f(1) - \int_0^1 \frac{x^2}{2} dx = \frac{1}{2} f(1) - \frac{1}{6}.$$

5. (a) Does the improper integral $\int_1^\infty \frac{1-\cos x}{x^3}\,dx$ converge?

Yes. $0 \leq \left|\frac{1-\cos x}{x^3}\right| \leq \frac{2}{x^3}$

and $\int_1^\infty \frac{2}{x^3}\,dx$ converges.

(b) Does the improper integral $\int_0^1 \frac{1-\cos x}{x^3}\,dx$ converge?

No

$0 \leq 1-\cos x = \frac{x^2}{2} + O(x^4)$

and so

$$\underbrace{\frac{1-\cos x}{x^3}}_{f(x)} = \underbrace{\frac{1}{x}}_{g(x)}\left(\frac{1}{2} + O(x)\right)$$

As $\frac{f(x)}{g(x)} \to \frac{1}{2}$ as $x \to 0$, and

$\int_0^1 g(x)\,dx$ diverges, so does $\int_0^1 f(x)\,dx$.