Math 125B, Winter 2015.
Mar. 4, 2015.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

1	
2	
3	
4	
TOTAL	

1. (a) Assume $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function. Assume $a \in \mathbb{R}^n$. State precisely what this means: f is differentiable at a.

There exists a linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{h \to 0} \frac{\| f(a+h) - f(a) - Th \|}{\| h \|} = 0.$$

(b) Assume $f : \mathbb{R}^2 \to \mathbb{R}$. Also assume that $f \in C^1(\mathbb{R}^2)$ and that $f(0,0) = 0$, $\frac{\partial f}{\partial x}(0,0) = 1$, and $\frac{\partial f}{\partial y}(0,0) = 2$. Compute $\lim_{n\to\infty} n \cdot f(1/n, 2/n)$.

Since $f \in \mathcal{C}^1(\mathbb{R})$, f is differentiable at all x, thus also at $(0,0)$, and $Df(0,0) = [1, 2]$.

Thus $\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f(0,0) - Df(0,0)\begin{bmatrix} x \\ y \end{bmatrix}}{\| (x,y) \|} = 0.$

Thus

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - x - 2y}{\sqrt{x^2+y^2}} = 0,$$

Plug in $x = \frac{1}{n}$, $y = \frac{2}{n}$.

$$\lim_{n\to\infty} \frac{f\left(\frac{1}{n}, \frac{2}{n}\right) - 5/n}{\frac{1}{n}\sqrt{5}} = 0$$

$$\lim_{n\to\infty} \left(n f\left(\frac{1}{n}, \frac{2}{n}\right) - 5 \right) = 0$$

__Answer__: 5

2. Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is given by:

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(a) Is f continuous on $\mathbb{R}^2$? $f \in \mathcal{C}^\infty(\mathbb{R}^2 \setminus \{0,0\})$, so the only issue is cont. and diff. at $(0,0)$.

Write $(x,y) = r(a,b)$, $a^2+b^2=1$ (so $|a|, |b| \le 1$)

$$f(x,y) = \frac{r^3 a^2 b}{r^2} = r a^2 b,$$

and so

$$|f(x,y)| \le r,$$

which is indep. of (a,b) and goes to 0 as $r \to 0$,

YES, f is continuous, as we have shown that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0).$$

(b) Is f differentiable on $\mathbb{R}^2$? (You do not need to compute partial derivatives away from $(0,0)$ to answer this question.)

$f(x,0) = 0$ for every x, so $f_x(0,0)=0$

$f(0,y) = 0$ for every y, so $f_y(0,0)=0$,

For differentiability at $(0,0)$, we thus need to check whether

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y)}{\sqrt{x^2+y^2}} = 0.$$

But, proceeding as in (a),

$$\frac{f(x,y)}{\sqrt{x^2+y^2}} = a^2 b,$$

and so the limit does not exist. NO, f

is not differentiable at $(0,0)$.

3. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is in $C^1(\mathbb{R}^2)$. Let $g : \mathbb{R}^3 \to \mathbb{R}$ be given by $g(x, y, z) = f(xy, y^2 + z^2)$.

(a) Show that $xz\frac{\partial g}{\partial x} - yz\frac{\partial g}{\partial y} + y^2\frac{\partial g}{\partial z} = 0$.

By C.R.: $\frac{\partial g}{\partial x} = f_x \cdot y$

$$\frac{\partial g}{\partial y} = f_x \cdot x + f_y \cdot 2y$$

$$\frac{\partial g}{\partial z} = f_y \cdot 2z$$

and so

$$xz\frac{\partial g}{\partial x} - yz\frac{\partial g}{\partial z} + y^2\frac{\partial g}{\partial z}$$

$$= xyz f_x - xyz f_x - 2y^2z f_y + 2y^2z f_y = 0.$$

(b) Assume $Df(1,1) = [-1 \quad 2]$. Compute $Dg(1,1,0)$.

By C.R.:

$$Df(1,1) = Df(1,1) \cdot Dh(1,1,0)$$

$$= [-1 \quad 2] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

$$= [-1 \quad 3 \quad 0]$$

Observe that, if $h(x,y,z) = (xy, y^2+z^2)$, then $g = f \circ h$

Also, $h(1,1,0) = (1,1)$.

$$Dh = \begin{bmatrix} y & x & 0 \\ 0 & 2y & 2z \end{bmatrix}$$

$$Dh(1,1,0) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

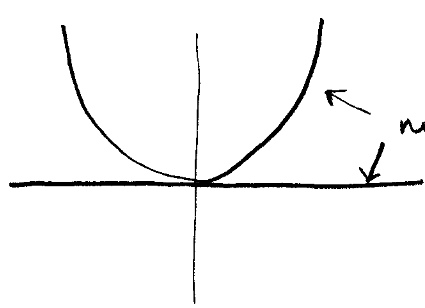
4. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x,y) = (x^2 + y, xy^2)$. Describe the set of points $(x,y) \in \mathbb{R}^2$ at which f has a differentiable local inverse.

$$Df = \begin{bmatrix} 2x & 1 \\ y^2 & 2xy \end{bmatrix}$$

$$\det Df = 4x^2y - y^2$$

$$= y(4x^2 - y) = 0 \quad \text{when } y = 0 \text{ or } y = 4x^2$$

The function has a differentiable local inverse everywhere but on pts in the set $\{(x,y) : y = 0 \text{ or } y = 4x^2\}$, so off these two graphs: $y = 0$, $y = 4x^2$



no diff. local inverse on these two graphs