Mar. 4, 2015.

MIDTERM EXAM 2

NAME(print in CAPITAL letters, first name first): ________________________________________

NAME(sign): ___________________________________________

ID#: ______________________________________

Instructions: Each of the 4 problems has equal worth. Read each question carefully and answer it in the space provided. You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise. Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 5 pages (including this one) with 4 problems.

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1. (a) Assume \( f : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a function. Assume \( a \in \mathbb{R}^n \). State precisely what this means: \( f \) is differentiable at \( a \).

There exists a linear map \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) so that

\[
\lim_{h \to 0} \frac{\| f(a + h) - f(a) - Th \|}{\| h \|} = 0.
\]

(b) Assume \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \). Also assume that \( f \in C^1(\mathbb{R}^2) \) and that \( f(0,0) = 0 \), \( \frac{\partial f}{\partial x}(0,0) = 1 \), and \( \frac{\partial f}{\partial y}(0,0) = 2 \). Compute \( \lim_{n \to \infty} n \cdot f(1/n, 2/n) \).

Since \( f \in C^1(\mathbb{R}) \), \( f \) is differentiable at all \( x \), thus also at \((0,0)\), and \( Df(0,0) = \begin{bmatrix} 1 & 2 \end{bmatrix} \).

Thus

\[
\lim_{(x,y) \to (0,0)} \frac{f(x,y) - f(0,0) - Df(0,0)[x,y]}{\| (x,y) \|} = 0.
\]

Thus

\[
\lim_{(x,y) \to (0,0)} f(x,y) = f(0,0) = 0.
\]

Thus

\[
\lim_{n \to \infty} \frac{f\left(\frac{1}{n}, \frac{2}{n}\right) - \frac{5}{n}}{\frac{1}{n} \sqrt{\frac{5}{n}}} = 0.
\]

\[
\lim_{n \to \infty} \left( n \cdot f\left(\frac{1}{n}, \frac{2}{n}\right) - 5 \right) = 0.
\]

Answer: 5
2. Assume \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is given by:

\[
f(x, y) = \begin{cases} \frac{x^2 + y^2}{2 + x^2 y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}
\]

(a) Is \( f \) continuous on \( \mathbb{R}^2 \)?

Write \((x, y) = r(a, b), a^2 + b^2 = 1 \) (so \(|a|, |b| \leq 1\))

\[
f(x, y) = \frac{r^2 a^2 b}{r^2} = r a^2 b,
\]
and so

\[
|f(x, y)| \leq r,
\]

which is independent of \( (a, b) \) and goes to 0 as \( r \to 0 \).

\( \text{YES} \) \( f \) is continuous, as we have shown that

\[
\lim_{(x, y) \to (0, 0)} f(x, y) = 0 = f(0, 0).
\]

(b) Is \( f \) differentiable on \( \mathbb{R}^2 \)? (You do not need to compute partial derivatives away from \( (0, 0) \) to answer this question.)

\[
f_x(x, 0) = 0 \quad \text{for every } x, \quad \text{so } f_x(0, 0) = 0,
\]

\[
f_y(0, y) = 0 \quad \text{for every } y, \quad \text{so } f_y(0, 0) = 0,
\]

For differentiability at \( (0, 0) \), we first need to check whether

\[
\lim_{(x, y) \to (0, 0)} \frac{f(x, y)}{\sqrt{x^2 + y^2}} = 0.
\]

But, proceeding as in (a),

\[
\frac{f(x, y)}{\sqrt{x^2 + y^2}} = a^2 b,
\]
and so the limit does not exist. \( \text{NO} \), \( f \)

is not differentiable at \( (0, 0) \).
3. Suppose \( f : \mathbb{R}^2 \to \mathbb{R} \) is in \( C^1(\mathbb{R}^2) \). Let \( g : \mathbb{R}^3 \to \mathbb{R} \) be given by \( g(x, y, z) = f(xy, y^2 + z^2) \).

(a) Show that \( xz \frac{\partial g}{\partial x} - yz \frac{\partial g}{\partial y} + y^2 \frac{\partial g}{\partial z} = 0 \).

By C.R.: \[ \frac{\partial g}{\partial x} = f_x \cdot y \]
\[ \frac{\partial g}{\partial y} = f_x \cdot x + f_y \cdot 2y \]
\[ \frac{\partial g}{\partial z} = f_y \cdot 2z \]
and so
\[ xz \frac{\partial g}{\partial x} - yz \frac{\partial g}{\partial y} + y^2 \frac{\partial g}{\partial z} = xy^2 f_x - xy^2 f_x - 2y^2 z f_y + 2y^2 z f_y = 0, \]

(b) Assume \( Df(1, 1) = [-1 \ 2] \). Compute \( Dg(1, 1, 0) \).

By C.R.: 
\[ Df(1, 1) = Df(1, 1) \cdot Dh(1, 1, 0) \]
\[ = [-1 \ 2] \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \]
\[ = [-1 \ 3 \ 0] \]

Observe that, if \( h(x, y, z) = (xy, y^2 + z^2) \), then \( g = f \circ h \). Also, \( h(1, 1, 0) = (1, 1) \). 
\[ Dh = \begin{bmatrix} y & 0 \\ 0 & 2y & 2z \end{bmatrix} \]
\[ Dh(1, 1, 0) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \]
4. Define \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by \( f(x,y) = (x^2 + y, xy^2) \). Describe the set of points \((x,y) \in \mathbb{R}^2\) at which \( f \) has a differentiable local inverse.

\[
\begin{bmatrix}
2x \\
y^2 \\
2xy
\end{bmatrix} =
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

\[
\det \begin{bmatrix}
2x \\
y^2 \\
2xy
\end{bmatrix} = 4x^2y - y^2 = y(4x^2 - y) = 0 \quad \text{when } y=0
\]

\[\quad \text{or } y = 4x^2\]

The function \( f \) has a differentiable local inverse everywhere but on points in the set \( \{ (x,y) : y=0 \text{ or } y=4x^2 \} \). So \( f \) has two graphs:

\[ y = 0, \ y = 4x^2 \]

no differentiable local inverse in these two graphs.