Math 127A, Fall 2019.

Discussion Problems 10

Note. The problems marked with (*) are part of the practice final. They will not be discussed in the last discussion session. The solutions to all problems are provided.

1. Assume $f : \mathbb{R} \to \mathbb{R}$ is a function.
(a) Show that f is continuous if and only if, for every closed set $F \subseteq \mathbb{R}$, $f^{-1}(F)$ is closed.
(b) Show that f is continuous if and only if, for every open set $G \subseteq \mathbb{R}$, $f^{-1}(G)$ is open.
(c) Assume f is continuous and $F \subseteq \mathbb{R}$ is closed. Is it necessarily true that $f(F)$ is closed?
(d)(*) Assume that f is continuous and $K \subseteq \mathbb{R}$ is compact. Is it necessarily true that $f(K)$ is compact?
(e)(*) Assume that f is continuous and $K \subseteq \mathbb{R}$ is compact. Is it necessarily true that $f^{-1}(K)$ is compact?
(f)(*) Assume that f is continuous and $A \subseteq \mathbb{R}$ is bounded. Is it necessarily true that $f(A)$ is bounded?
(g)(*) Assume that $f(K)$ is compact for every $K \subseteq \mathbb{R}$. Is f necessarily continuous?
(h)(*) Assume that f is continuous. Show that its set of zeros, $\{x \in \mathbb{R} : f(x) = 0\}$ is closed.

2. Let $F \subseteq \mathbb{R}$ be a nonempty closed set and let $g(x) = \inf\{|x - z| : z \in F\}$ be the distance of x to F. Show that g is uniformly continuous, and is nonzero on F^c.

3. Show that if $f : A \to \mathbb{R}$ is uniformly continuous, and set A is bounded, then its range $f(A)$ is bounded. Is this still true if f is merely continuous? (Note that this in not the same question as 1(f), as f is not defined and continuous on $\mathbb{R}$!)

4. (a)(*) Prove that $f(x) = \sqrt{x^2 + 1}$ is uniformly continuous on $(0, \infty)$.
(b) Prove that $f(x) = x\sqrt{x}$ is not uniformly continuous on $[0, \infty)$.

5. Problem 4.5.2 in the book.

6. Problem 4.5.6 (a) in the book.

7(*). (a) Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show that there exists an $x \in [0, 1]$ so that $f(x) = x$. (b) Is the conclusion in (a) true if f is a continuous function on the *open* interval $(0, 1)$ with range also contained in $(0, 1)$? (c) Is the conclusion in (a) true if f is a continuous *decreasing* function on the *open* interval $(0, 1)$ with range also contained in $(0, 1)$?

1. (a) ($\Longrightarrow$) Assume F is closed. Assume $x_n \in f^{-1}(F)$, and $x_n \to x$. Then $f(x_n) \in F$ (by definition), and $f(x_n) \to f(x)$ (by continuity), and finally $f(x) \in F$ (as F is closed). Then $x \in f^{-1}(F)$. It follows that $f^{-1}(F)$ is closed.
($\Longleftarrow$) Now assume that f is not continuous, that is, there exists a sequence $x_n \to x$ so that $f(x) \not\to f(x)$. By passing to a subsequence, we may assume that there is an $\epsilon > 0$ so that $|f(x_n) - f(x)| \geq \epsilon$. Consider the set $F = V_\epsilon(f(x))^c$. This is a closed set, as the complement of open set $V_\epsilon(f(x))$. Now, as $f(x_n) \notin V_\epsilon(f(x))$, and so $x_n \in f^{-1}(F)$. But clearly $f(x) \notin F$, so $x \notin f^{-1}(F)$. So, $f^{-1}(F)$ is not closed. We have found a closed set whose preimage $f^{-1}(F)$ is not closed.
(b) This follows from (a) and the fact that $f^{-1}(F^c) = (f^{-1}(F))^c$.
(c) No. Assume $F = [0, \infty)$, and $f(x) = 1/(1 + |x|)$. This function maps $[0, \infty)$ to $(0, 1]$.
(d) Yes. This follows from the theorem we proved is class, as a continuous function on $\mathbb{R}$ is also continuous as a function on any subset of $\mathbb{R}$, thus in particular on K. We can of course also reproduce the proof, which we do here. Assume that $y_n \in f(K)$. Then $y_n = f(x_n)$, $x_n \in K$, and so there is a subsequence x_{n_k} converging to $x \in K$. Then, by continuity, $y_{n_k} = f(x_{n_k}) \to f(x) \in f(K)$. We have found a subsequence of (y_n) with a limit in $f(K)$.
(e) No. Assume $f(x) = 0$ for all x, i.e., f is the constant zero function. Then $\{0\}$ is compact, but $f^{-1}(\{0\}) = \mathbb{R}$ is not compact.
(f) If A is bounded, $A \subseteq [-M, M]$ for some $M \geq 0$. Then $f(A) \subseteq f(-M, M])$, and $f([-M, M])$ is compact (by (d)), thus bounded.
(g) Assume that

$$f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Then $f(K)$ is finite thus compact for *every* set K, but f is not continuous.
(h) This follows from (a) The set of zeros is the preimage $f^{-1}(\{0\})$, and $\{0\}$ is closed.

2. We first prove that f is nonzero on F^c. If $g(x) = 0$, there must exist a sequence $z_n \in F$, so that $|x - z_n| < 1/n$. But this means that $z_n \to x$, and, as F is closed, $x \in F$.

Pick an $\epsilon > 0$. We claim that if $|x - y| < \epsilon/2$, then $|g(x) - g(y)| < \epsilon$. Find a $z_1 \in F$ so that $|x - z_1| \leq g(x) + \epsilon/2$ and a $z_2 \in F$ so that $|y - z_2| < g(y) + \epsilon/2$. Then

$$g(x) \leq |x - z_2| \leq |x - y| + |y - z_2| \leq \epsilon/2 + g(y) + \epsilon/2 = g(y) + \epsilon.$$

By symmetry also

$$g(y) \leq g(x) + \epsilon$$

and so $|g(x) - g(y)| < \epsilon$.

3. We know from class that continuity is not enough: for example, $f(x) = 1/x$ on $(0, 1)$ is not bounded.

Now assume that f is uniformly continuous. Pick $\epsilon = 1$ and find a δ so that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. Then, for any $a \in \mathbb{R}$, and any interval $[a, a + \delta)$, there exists a constant M_a so that $|f(x)| \leq M_a$ for every $x \in [a, a + \delta) \cap A$. Indeed, either $[a, a + \delta) \cap A = \emptyset$ (in which case we can take $M_a = 0$), or there exists an $x_0 \in [a, a + \delta) \cap A$, and then for every $x \in [a, a + \delta) \cap A$, $|f(x)| \leq |f(x_0)| + |f(x) - f(x_0)| \leq |f(x_0)| + 1$ and so we can take $M_a = |f(x_0)| + 1$.

As A is bounded, there exits an $N > 0$ so that $A \subset [-N, N]$, We can cover $[-N, N]$ with finitely many intervals $[a_i, a_i + \delta)$, $i = 1, \ldots n$, by putting them side-by-side (i.e., $a_i = -N + i\delta$), and then $f(x) \leq \max\{M_{a_i} : i = 1, \ldots, n\}$.

4. (a) By algebra,

$$f(x) - f(y) = \frac{(x - y)(x + y)}{\sqrt{x^2 + 1} + \sqrt{y^2 + 1}},$$

and then the inequalities $|x| \le \sqrt{x^2+1}$ and $|y| \le \sqrt{y^2+1}$ imply that

$$|f(x) - f(y)| \le |x-y| \cdot \left(\frac{|x|}{\sqrt{x^2+1}} + \frac{|y|}{\sqrt{y^2+1}} \right) \le 2|x-y|.$$

Uniform continuity follows: given an $\epsilon > 0$, we may take $\delta = \epsilon/2$, as then $|x-y| < \delta$ implies $f(x) - f(y)| < \epsilon$.
(b) Let $x_n = n + 1/\sqrt{n}$, $y_n = n$. Then $x_n - y_n = 1/\sqrt{n} \to 0$. Moreover,

$$\begin{aligned} f(x_n) - f(y_n) &= n(\sqrt{n+1/\sqrt{n}} - \sqrt{n}) + \frac{1}{\sqrt{n}}\sqrt{n+1/\sqrt{n}} \\ &= n\frac{n+1/\sqrt{n}-n}{\sqrt{n+1/\sqrt{n}}+\sqrt{n}} + \frac{1}{\sqrt{n}}\sqrt{n+1/\sqrt{n}} \\ &= \frac{\sqrt{n}}{\sqrt{n+1/\sqrt{n}}+\sqrt{n}} + \frac{1}{\sqrt{n}}\sqrt{n+1/\sqrt{n}} \\ &\to \frac{1}{2} + 1 = \frac{3}{2}, \end{aligned}$$

proving that f is not uniformly continuous by a theorem from the lecture.

5. (a) For example, $f(x) = \sin x$ on $(-2\pi, 2\pi)$. (We assume here that it is known that $\sin x$ is a continuous function on $\mathbb{R}$ that satisfies the properties from trigonometry.)
(b) Impossible. The range of a continuous function on a closed interval is a closed interval, by the intermediate value theorem.
(c) Let $f = 1/(1-x^2)$ on $(-1,1)$. The range is $[1,\infty)$.
(d) Impossible. By the intermediate value theorem, if every rational number is in the range of f, all real numbers are in the range of f.

6. Denote $a = f(0) = f(1)$. Define the function $g : [0,1/2] \to \mathbb{R}$ by $g(x) = f(x+1/2) - f(x)$. Then $g(0) = f(1/2) - a$ amd $g(1/2) = a - f(1/2) = -g(0)$. So either $g(0)$ and $g(1/2)$ have the opposite sign or they are both 0. By the intermediate value theorem, there exists an $x \in [0,1/2]$ so that $g(x) = 0$. This means that $f(x+1/2) = f(x)$ and we can take $y = x+1/2$ to get the desired x and y.

7. (a) Let $g : [0,1] \to \mathbb{R}$ be defined by $g(x) = f(x) - x$. Then g is continuous, as the difference between two continuous functions, $g(0) = f(0) \ge 0$ and $g(1) = f(1) - 1 \le 0$. By intermediate value theorem, there exists an $x \in [0,1]$ so that $g(x) = 0$. This means that $f(x) = x$.
(b) No. Take the function $f(x) = x^2$.
(c) Yes. Let $g : (0,1) \to \mathbb{R}$ be again defined by $g(x) = f(x) - x$. Then $\lim_{x\to 0} g(x) = \lim_{x \to 0} f(x) > 0$ (note that this is the same as the right limit at 0, as 0 is the left endpoint of the domain of f) and $\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) - 1 < 0$ (this time, this is the same as the left limit at 1). Therefore there exist $x_1, x_2 \in (0,1)$ so that $x_1 < x_2$ and $g(x_1) > 0$ and $g(x_2) < 0$. Then, by the intermediate value theorem, there exists an $x \in [x_1, x_2]$ so that $g(x) = 0$.