Math 127A, Fall 2019.

**Discussion problems 2**

1. (a) Show that for every real numbers $a, b > 0$, their geometric mean is no larger than their arithmetic mean, that is, $\sqrt{ab} \leq \frac{a+b}{2}$.
(b) (*Optional*[1].) One can get a lot of very sophisticated inequalities out of the order field axioms; here is one from a math competition. Compute the minimum of the set

$$\{(1+x_1)(1+x_2)(1+x_3) : x_1, x_2, x_3 \geq 0, x_1x_2x_3 \geq 1\}.$$

(*Hints*. Argue that you may assume $x_1x_2x_3 = 1$. Then replace both $x_1$ and $x_2$ by $x_1' = x_2' = \sqrt{x_1x_2}$. Show that $(1+x_1')(1+x_2') \leq (1+x_1)(1+x_2)$. Argue that this means that the minimum is achieved when all $x_i$ are equal.)

2. Find the supremum of $A$, if it exists, and the infimum of $A$, if it exists, in each case below. You do not need to prove your assertions.
(a) $A = \mathbb{N}$.
(b) $A = \{\frac{7}{n} : n \in \mathbb{N}\}$.
(c) $A = \{\frac{n}{n+1} : n \in \mathbb{N}\}$.
(d) $A = (1, 3) \cup [5, 7)$.
(e) $A = \cup_{n=1}^{\infty}(n, n + 1/n)$.
(f) $A = \{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$.
(g) $A = \{1 - r^2 : r \in \mathbb{Q}\}$.

3. Suppose $A \subseteq \mathbb{R}$ and $B \subseteq R$ are both nonempty. Suppose also that $a < b$ for all $a \in A$ and $b \in B$.
(a) Prove that $\sup A \leq \inf B$.
(b) Must it be true that $\sup A < \inf B$?

4. Does the set $[0, \sqrt{2}] \cap \mathbb{Q}$ have a minimum? A maximum?

5. Prove that, for $A \subset \mathbb{R}$, $\inf A = \sup A$ if and only if $A$ is a singleton (that is, if and only if $A = \{a\}$ for some $a \in \mathbb{R}$).

6. Prove that for every $a, b \in \mathbb{R}$, $||a| - |b|| \leq |a - b|$ and $|a^2 - b^2| \leq (|a| + |b|)|a - b|$.

7. Find the minimum of the set

$$\{|x| + |x-1| + |x-3| : x \in \mathbb{R}\}.$$

[1] Optional problems will not be covered in discussions and you will not be required to know how to do them.