Math 127A, Fall 2019.

Discussion problems 4

Note. Some of these problems were taken from exams at universities in Slovenia and Croatia, in which case the location is given in brackets. Some of these problems are optional; take them as a personal challenge. The formulations are translated from Slovenian or Croatian with no changes. These problems may look tough, as the students are apparently required to see an algebraic transformation that makes the problem clearer.

1. Assume that the sequence (a_n) is such that the set $A = \{a_n : n \in \mathbb{N}\}$ is bounded above.

(a) Is it necessarily true that (a_n) has convergent subsequence?

(b) Assume in addition that the set A does not have a maximum, that is, that max A does not exist. Show that then (a_n) has convergent subsequence.

2. First do problem 2.4.7 in the book (which is assigned for homework).

(a) Assume that (a_n) is a bounded sequence. Prove that $\limsup a_n \leq b$ if and only if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that $n \geq N$ implies $a_n < b + \epsilon$.

(b) Assume that (a_n) is a bounded sequence and $a = \limsup a_n$. Prove that there exists a subsequence of (a_n) that converges to a.

(c) Assume that (a_n) is a bounded sequence and $a = \limsup a_n$. Prove that the limit of any convergent subsequence of (a_n) is less or equal to a.

(d) Assume that (a_n) and (b_n) are bounded sequences. Prove that $\limsup a_n + b_n \le \limsup a_n + \limsup b_n$, but that equality does not always hold.

(e) Assume that (a_n) is a convergent sequence and (b_n) is a bounded sequence. Prove that $\limsup_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$.

(f)[Zagreb] Optional. Prove or disprove the following claim. If a bounded sequence (a_n) of real numbers is such that $\limsup (a_n + b_n) = \limsup a_n + \limsup b_n$ for every bounded sequence (b_n) , then (a_n) converges.

(g) Formulate the analogous properties to (a)–(f) for liminf.

3. A sequence (x_n) is given by $x_1 = 1$ and

$$x_{n+1} = \sqrt{1 + x_n}$$

for $n \geq 1$.

(a) Prove that the sequence is increasing and bounded above by 2.

(b) Find the limit of the sequence.

4. [Ljubljana] A sequence (x_n) is given by $x_1 = 1$ and

$$x_{n+1} = \frac{4x_n - 1}{x_n + 1}$$

for $n \ge 1$. Prove that the sequence is monotone and bounded and find its limit.

5. [Zagreb] Optional. A sequence (b_n) of real numbers is given recursively by

$$b_1 = 0,$$
 $b_{n+1} = \frac{3 - b_n^2}{2}, n \ge 1.$

- (a) Show that the sequence (b_n) is not monotone, even starting from some term.
- (b) Prove that the sequence (b_n) nevertheless converges and compute its limit.
- 6. [Zagreb] Optional. A sequence (b_n) of real numbers is given recursively by

$$b_1 = \frac{1}{3}, \qquad 3b_{n+1} = b_1b_1\cdots b_n + 2, \ n \ge 1.$$

- (a) Prove that $b_n < 1$ for every $n \in \mathbb{N}$.
- (b) Prove that the sequence (b_n) converges and determine its limit.

7. [Zagreb] Optional. Depending on the parameter $\alpha > 0$, investigate convergence of the sequence (a_n) given recursively by

$$a_1 = \frac{\alpha}{2}, \qquad a_{n+1} = \frac{1}{2}(\alpha + a_n^2), \ n \in \mathbb{N}.$$

In case of convergence, determine $\lim_{n \to \infty} a_n$.

8. Compute $\liminf x_n$ and $\limsup x_n$ for the sequences

(a)
$$x_n = 2 + (-1)^n n^{-1}$$
, (b) $x_n = n^{(-1)^n} + n^{-1}$, (c) $x_n = \frac{(-1)^n n + (-1)^{\lfloor n/2 \rfloor} n + 7(-1)^n}{n}$.