Math 25, Fall 2014.

Discussion problems 4 Solutions

1. (a) No. Take $a_n = -n$ which diverges to $-\infty$ and so does any subsequence.
(b) Let $a = \sup A$. For any $b < a$, there is an n such that $a_n > b$ (by definition of $\sup A$), and such that $a_n < a$ (as $\max A$ does not exist). Choose n_1 so that $a_{n_1} \in (a-1, a)$. Then recursively define n_k so that $n_{k+1} > n_k$, $a_{n_{k+1}} > a_{n_k}$, and $a_{n_k} \in (a - 1/k, a)$. As $a - 1/k < a_{n_k} < a$, $a_{n_k} \to a$ by order theorems.

2. (a) ($\Longrightarrow$) Let $a = \limsup a_n$ and pick an $\epsilon > 0$. Then (by the theorem from lecture), there is an $N \in \mathbb{N}$ so that $n \geq N$ implies $a_n < a + \epsilon$. As $a \leq b$, this implies $a_n \leq b + \epsilon$.
($\Longleftarrow$) By the order theorem for lim sup, for any $\epsilon > 0$, $\limsup a_n \leq b + \epsilon$. Thus $\limsup a_n \leq b$.

(b) Choose n_1 so that $a_{n_1} \in (a-1, a+1)$. Then recursively define n_k so that $n_{k+1} > n_k$, and $a_{n_k} \in (a - 1/k, a + 1/k)$. This is possible as there are infinitely many terms of the sequence in $(a - 1/k, a + 1/k)$. As $a - 1/k < a_{n_k} < a + 1/k$, $a_{n_k} \to a$ by order theorems.

(c) Pick a convergent subsequence (a_{n_k}) and arbitrary $\epsilon > 0$. For all but finitely many k, $a_{n_k} \leq a + \epsilon$. Then by order theorems $\lim_k a_{n_k} \leq a + \epsilon$. As this holds for arbitrary $\epsilon > 0$, $\lim_k a_{n_k} \leq a$.

(d) Let $a = \limsup a_n$ and $b = \limsup b_n$, and pick any $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ so that, for $n \geq N$, $a_n < a + \epsilon$ and $b_n < b + \epsilon$. Then, for $n \geq N$, $a_n + b_n < a + b + \epsilon$. By order theorems, $\limsup(a_n + b_n) \leq a + b + \epsilon$. As this holds for arbitrary $\epsilon > 0$, $\limsup(a_n + b_n) \leq a + b$.

Take $(a_n) = (1, 0, 1, 0, \ldots)$ and $(b_n) = (0, 1, 0, 1, \ldots)$. Then $\limsup a_n = \limsup b_n = 1$ and $\limsup(a_n + b_n) = 1 \neq 2 = \limsup a_n + \limsup b_n$.

(e) By (d), we have $\limsup(a_n + b_n) \leq \lim a_n + \limsup b_n$. For the reverse inequality, write $b_n = (a_n + b_n) - a_n$ and use (d) again to get $\limsup b_n \leq \limsup(a_n + b_n) + \limsup(-a_n)$. But $-a_n$ converges with limit $-\lim a_n$, which is then also its lim sup. Thus $\limsup b_n \leq \limsup(a_n + b_n) - \lim a_n$ and $\limsup b_n + \lim a_n \leq \limsup(a_n + b_n)$. The two inequalities prove the claimed equality.

(f) The statement is true and we will prove the converse: if $L = \limsup a_n$ and $\ell = \liminf a_n$ and $\ell < L$, then there exists a bounded sequence b_n so that $\limsup(a_n + b_n) \neq \limsup a_n + \limsup b_n$. Assume that a_{n_k} is a subsequence that converges to ℓ, and define b_n to be $L - \ell$ if $n = n_k$ for some k and 0 otherwise. Then $\limsup b_n = L - \ell$, $\limsup(a_n + b_n) = L$, and $\limsup a_n + \limsup b_n = 2L - \ell > L$.

(g) First we observe that

$$\liminf(a_n) = -\limsup(-a_n),$$

which follows by definition and can be used to easily prove all properties below using the already proved facts about lim sup.
(a') We have $\liminf a_n \geq b$ if and only if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that $n \geq N$ implies $a_n > b - \epsilon$.
(b') Let $\alpha = \liminf a_n$. There exists a subsequence of (a_n) that converges to α.
(c') Let $\alpha = \liminf a_n$. The limit of any convergent subsequence of (a_n) is greater or equal to α.
(d') We have $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$, but equality does not always hold.
(e') Assume that (a_n) is a convergent sequence and (b_n) is a bounded sequence. Then $\liminf(a_n + b_n) = \lim a_n + \liminf b_n$.

(f') If a bounded sequence (a_n) of real numbers is such that $\liminf(a_n+b_n) = \liminf a_n + \liminf b_n$ for every bounded sequence (b_n), then (a_n) converges.

3. (a) Clearly $x_1 < 2$. If $x_n < 2$, then $x_{n+1} < \sqrt{1+2} = \sqrt{3} < 2$. Thus the boundedness claim follows by induction.

Next we prove that $x_n < x_{n+1}$ for all n. This is true for $n=1$ as $x_2 = \sqrt{2}$. Assuming $x_n < x_{n+1}$, $1+x_n < 1+x_{n+1}$ and then $\sqrt{1+x_n} < \sqrt{1+x_{n+1}}$, that is, $x_{n+1} < x_{n+2}$. By induction, the sequence is increasing.

(b) As the sequence is monotone and bounded, $x = \lim x_n$ exists. Then $1+x_n \to 1+x$ by algebraic limit theorem, and then $\sqrt{1+x_n} \to \sqrt{1+x}$ by the homework problem 2.3.1. Therefore, x must satisfy $x = \sqrt{1+x}$, $x^2 - x - 1 = 0$. As $x_n \geq 1$ for all n, $x \geq 1$. The only solution of the quadratic equation with $x \geq 1$ is $x = (1+\sqrt{5})/2$, the golden ratio, which is the answer.

4. Rewrite

$$x_{n+1} = 4 - \frac{5}{x_n+1}.$$

Note that $x_2 = 3/2 > x_1$ and, by standard induction (as in problem 3), x_n is increasing. Also $x_n \leq 4$ implies $x_{n+1} \leq 3$, so by induction the sequence is bounded above by 4. Then $x = \lim x_n$ exists, and satisfies $x \geq 1$ and $x^2 - 3x + 1 = 0$, thus $x = (3+\sqrt{5})/2$.

5. Let $a_n = 1 - b_n$. Then $a_1 = 1$ and $a_{n+1} = \frac{1}{2}a_n(a_n - 2)$. If $a_n \in (0,1]$, then $a_{n+1} \in [-1/2, 0)$, while if $a_n \in [-1/2, 0)$, then $a_{n+1} \in (0,1]$. By induction, a_n is bounded and changes sign at every step, thus a_n never becomes monotone (and neither does b_n). Moreover, if $a_n > 0$, then $|a_{n+1}| = -a_{n+1} = a_n - \frac{1}{2}a_n^2 < a_n$. Also, if $a_n > 0$, $a_{n+2} = a_n + \frac{1}{8}a_n^3(a_n - 4)$ (after some algebra), so $a_{n+2} < a_n$. It follows that the subsequence of positive terms a_{2k+1} converges to limit $a \in [0,1]$ which satisfies $a^3(4-a) = 0$, thus $a = 0$. Then the subsequence of negative terms converges to 0 as well. Thus $\lim a_n = 1$.

6. (a) The fact that $0 < b_n < 1$ is easily proved by induction. (b) Let $y_n = b_1 \cdots b_n$. We have $y_1 = 1/3$ and $y_{n+1} = \frac{1}{3}y_n^2 + \frac{2}{3}y_n$. Also, $0 < y_n < 1$ by (a). Then $y_n^2 < y_n$, so $y_{n+1} < y_n$. Thus $y = \lim y_n$ exists and $y < 1$. Also $y^2 - y = 0$, so $y = 0$. As $3b_{n+1} = y_n + 2$, $\lim b_n = 2/3$.

7. By induction, $a_n > 0$ for every n. Moreover, $a_2 > a_1$ and then by standard induction $a_{n+1} > a_n$ for all n. Thus the sequence is increasing. If it is bounded, then it must converge to a limit, say a, satisfying $a^2 - 2a + \alpha = 0$. If $\alpha > 1$, this equation has no real solution, so the sequence must be unbounded.

Assume now that $\alpha \leq 1$. We prove by induction that $a_n \leq 1 + \sqrt{1-\alpha}$ for all n. For $n = 1$ this holds as $\alpha/2 \leq 1 + \sqrt{1-\alpha}$. Assuming the validity for n, it is a simple check that $a_{n+1} = \frac{1}{2}(\alpha + a_n^2) \leq 1 + \sqrt{1-\alpha}$. Therefore (a_n) is bounded and thus converges to a limit $a > \alpha/2$. As $\alpha/2 > 1 - \sqrt{1-\alpha}$, $a = 1 + \sqrt{1-\alpha}$.

8. (a) As $-1/n \leq (-1)^n n^{-1} \leq 1/n$, the sandwich theorem implies that $\lim x_n = 2$, and so $\liminf x_n = \limsup x_n = 2$.

(b) For even $n = 2k$, $x_{2k} = 2k + 1/(2k) \to \infty$. For odd $n = 2k+1$, $x_{2k} = 2/(2k+1) \to 0$. It follows that $\limsup x_n = \infty$ and $\liminf x_n = 0$.

(c) If n is divisible by 4, i.e., $n = 4k$ for some integer $k \geq 0$, $x_n = 2 + 7/n \to 2$. When $n = 4k+1$, $x_n = -7/n \to 0$. When $n = 4k+2$, $x_n = 7/n \to 0$. When $n = 4k+3$, $x_n = -2 - 7/n \to -2$. Therefore, $\liminf x_n = -2$ and $\limsup x_n = 2$.