Math 127A, Fall 2019.

Discussion problems 5

Note. These problems are also practice exam for Midterm 1. They are a bit longer than the actual exam, which will otherwise be very similar. My advice is to give yourself 60 minutes and no distractions to write out the solutions *before the discussion on Thursday*. Problems 1–4 and 5(a) are from the sample midterms 1 and 2, where you can find the solutions. Solutions to Problems 5(b), 6, and 7 are on page 3.

1. (a) Assume (a_n) is a sequence of real numbers, and $a \in \mathbb{R}$. Define precisely what this statement means: $\lim\limits_{n\to\infty} a_n = a$.
(b) Assume (a_n) is a sequence of real numbers. Define precisely what this statement means: (a_n) is bounded.
(c) Let $a_n = \dfrac{2\sqrt{n}+1}{2\sqrt{n}-3}$. Using only the definition in (a), prove that that $\lim\limits_{n\to\infty} a_n = a$ for some $a \in \mathbb{R}$. (Identify a first.)
(d) Is the sequence (a_n) from (c) bounded?

2. (a) Recall that $n! = 1 \cdot 2 \cdots n$ is the product of first n natural numbers. Prove by induction that for all $n \in \mathbb{N}$, $n \geq 4$ implies that $2^n < 3(n-1)!$.
(b) Use (a), or any other method, to prove that $\lim\limits_{n\to\infty} \dfrac{2^n}{n!} = 0$.

3. Find

$$\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 2 + \frac{1}{n}\right).$$

Carefully prove your assertion.

4. *Do not consider extended real line in this problem, so no ∞ or $-\infty$.*
(a) State the definition of $\sup A$ for a set $A \subset \mathbb{R}$.
(b) What do you need to assume about a set $A \subset \mathbb{R}$ to ensure that $\sup A$ exists?
(c) Let $A = (0, 2)$ and $B = A \cap \mathbb{Q}$. Find $\sup A$ and $\sup B$. Carefully prove your assertions.
(d) Assume that $A \subseteq \mathbb{R}$. Assume also that A is bounded above and that $A \cap \mathbb{Q}$ is nonempty. Is it necessarily true that $\sup A = \sup(A \cap \mathbb{Q})$?

5. (a) Assume $x_1 = 1$ and

$$x_{n+1} = \sqrt{6 + x_n} \qquad \text{for } n \in \mathbb{N}.$$

Show that the sequence is increasing, that $x_n \leq 3$ for all $n \in \mathbb{N}$, and that $\lim_{n\to\infty} x_n$ exists. Compute the limit.
(b) Assume $x_1 = 1$ and

$$x_{n+1} = \sqrt{x_n + x_n^2} \qquad \text{for } n \in \mathbb{N}.$$

Show that the sequence is increasing and that $\lim_{n\to\infty} x_n = \infty$.

6. (a) Assume $a_n > 0$. Show that $\lim a_n = \infty$ if and only if $\lim 1/a_n = 0$.

(b) Use (a) and algebraic limit theorem to show that $\lim \dfrac{n^5 + n - 100}{n^4 + n^2 + 1} = \infty$.

7. Let $x_n = \dfrac{(-1)^n n^2 - n^2 + 3n}{n^2 + 3}$. Compute $\limsup x_n$ and $\liminf x_n$. Does (x_n) converge? Is it bounded? Does it have a convergent subsequence?

Solution to 5(b). We prove by induction that $x_n \geq 1$: this is true for $n = 0$ and the $n \to n+1$ step follows from $x_{n+1} \geq \sqrt{1+1} = \sqrt{2} > 1$. As $x_{n+1} = \sqrt{x_n + x_n^2} > \sqrt{x_n^2} = x_n$, the sequence is increasing. If it were bounded, then it would have a limit x, which would satisfy $x = \sqrt{x + x^2}$, $x^2 = x + x^2$, $x = 0$. But, by the order theorem, $x \geq 1$. From this contradiction we conclude that the limit does not exist, and so (x_n) is unbounded. As it is increasing, it must diverge to ∞.

Solution to 6. (a) ($\Longrightarrow$) Pick an $\epsilon > 0$. As $a_n \to \infty$, there is an $N \in \mathbb{N}$ so that $n \geq N$ implies $a_n \geq 1/\epsilon$. Then, for $n \geq N$,

$$0 < 1/a_n < \epsilon,$$

and so $|1/a_n - 0| = |1/a_n| < \epsilon$. We have shown that $1/a_n \to 0$.
($\Longleftarrow$) Assume $a_n \to \infty$. Pick an $M > 0$. As $1/a_n \to 0$, there is an $N \in \mathbb{N}$ so that $n \geq N$ implies $1/a_n < 1/M$. Then, for $n \geq N$, $a_n > M$. We have shown that $a_n \to \infty$.
(b) Let $a_n = \dfrac{n^5 + n - 100}{n^4 + n^2 + 1}$. Then $a_n > 0$ for $n \geq 100$, so we can apply (a). Moreover, we can write

$$a_n = \frac{1 + 1/n^4 - 100/n^5}{1/n + 1/n^3 + 1/n^5},$$

so clearly $1/a_n \to 0$ by the algebraic limit theorem (and the fact that $1/n \to 0$).

Solution to 7. For odd n, $x_n = \dfrac{-2n^2 + 3n}{n^2 + 3} = \dfrac{-2 + 3/n}{1 + 3/n^2} \to -2$. For even n, $x_n = \dfrac{3n}{n^2 + 3} = \dfrac{3/n}{1 + 3/n^2} \to 0$. Therefore, $\limsup x_n = 0$ and $\liminf x_n = -2$. As these two are different, the sequence does not converge. As they are both finite, the sequence is bounded. We have exhibited *two* convergent subsequences.