Math 127A, Fall 2019.

Discussion problems 6

Note. Justify the answers to all yes-no questions with a proof or with a counterexample. All proofs in this set are very short.

1. (a) Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

converges and find its sum.
(b) Is there a rearrangement of the series in (a) with a different sum?

2. Do the exercise 2.7.7(*) first. This is not in the book, but assigned separately on the homework page, and it asks you to prove the following: If $\sum_n x_n$ converges absolutely and (y_n) is bounded, then $\sum_n x_n y_n$ converges absolutely.

Assume (a_n) and (b_n) are two sequences such that $a_n > 0$ and $b_n > 0$ for every n, and such that $\lim_{n\to\infty} b_n/a_n$ exists and is nonzero. Prove that $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges. (This is known as the *Limit Comparison Test* in calculus).

3. For each of the following series, determine (with proof) whether it converges absolutely, converges conditionally, or diverges:
(a) $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k(k+1)}}$
(b) $\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k(k+1)}}$
(c) $\sum_{k=1}^{\infty} (-1)^k \frac{1}{\sqrt{k(k^3+1)}}$
(d) $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \dots$ (Every third term is negative.)

4. Assume $a_k > 0$ for all $k \in \mathbb{N}$. For each statement below, prove it or find a counterexample.
(a) If $\sum_{k=1}^{\infty} a_k$ converges, then $\sum_{k=1}^{\infty} 1/a_k$ diverges.
(b) If $\sum_{k=1}^{\infty} a_k$ converges, then (a_k) is a decreasing sequence.
(c) If $\limsup_n na_n > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.
(d) If $\liminf_n na_n > 1$, then $\sum_{k=1}^{\infty} a_k$ diverges.
(e) If $\lim a_n = 0$, then $\sum_{k=1}^{\infty} (-1)^k a_k$ converges.