Math 127A, Fall 2019.

Discussion Problems 9

Note. These problems are also practice exam for Midterm 2. They are a bit longer than the actual exam, which will otherwise be similar. My advice is to give yourself 60 minutes and no distractions to write out the solutions *before the discussion on Thursday*. Problems 1(a–d), 2, 3(a) are problems 2, 3, 4(b) from the sample midterm 2, where you can find the solutions. Solutions to Problems 1(e–g), 3(b), 4, and 5 are on page 3.

1. For part (a), you may use without definition the concept of *limit of a sequence*. Then, for each of the series in (b), (c), (d), determine (with proof) whether it converges absolutely, converges conditionally, or diverges.

(a) Assume a_n, $n \in \mathbb{N}$, are real numbers. Define precisely what these two statements mean: $\sum_{k=1}^{\infty} a_k$ converges absolutely; $\sum_{k=1}^{\infty} a_k$ converges conditionally.

(b) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k} + \frac{7}{k!}\right)$

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{3k}{4k+1}\right)^k$

(d) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{\sqrt{k}}$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n!}}$

(f) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2+1}$

(g) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-1}{n} - \frac{n}{n+1}\right)$

2. Assume that $a_k > 0$ for all $k \in \mathbb{N}$. For each statement below, prove it or find a counterexample.

(a) If $a_k = 1$ for all even k, then $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges.

(b) If $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, then $\displaystyle\sum_{k=1}^{\infty} (a_k + a_k^2)$ converges.

(c) If $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges, then $\displaystyle\sum_{k=1}^{\infty} a_k^2$ converges.

3. (a) Prove: if $\liminf(na_n) = 2$, then $\sum_{n=1}^{\infty} a_n$ is a divergent series.
(b) True or false: if $\limsup(na_n) = 2$, then $\sum_{n=1}^{\infty} a_n$ is a divergent series.

4. In all parts, fix an $r > 0$. (a) Assume that $A \subseteq \mathbb{R}$ is arbitrary. Show that the set

$$G = \{x \in \mathbb{R} : |x - a| < r \text{ for some } a \in A\}$$

is open.
(b) Assume that $A \subseteq \mathbb{R}$ is arbitrary. Is the set

$$B = \{x \in \mathbb{R} : |x - a| \leq r \text{ for some } a \in A\}$$

necessarily closed?
(c) Assume that $A \subseteq \mathbb{R}$ is closed. Prove that the set B in (b) is closed.
(d) Assume that $A \subseteq \mathbb{R}$ is compact. Prove that the B in (b) is compact.

5. For each of the following statements determine, with proof, whether it is true or false.
(a) If $A \subseteq \mathbb{R}$ is connected, then A^c is connected.
(b) If $A \subseteq \mathbb{R}$ is connected, then A^c is disconnected.
(c) If $A \subseteq \mathbb{R}$ is closed then $A \cap [0, 1]$ is compact.
(d) The set $\{(-1)^n/n : n \in \mathbb{N}\}$ is closed.
(e) If $A \subseteq \mathbb{R}$ is bounded, its closure $\overline{A}$ is also bounded.
(f) If a set $A \subseteq \mathbb{R}$ is unbounded, its interior A° is also unbounded.

1. We will denote by a_n the nth term in each case.
(e) Using the ratio test, we get

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}\sqrt{n!}}{2^n\sqrt{(n+1)!}} = \frac{2}{\sqrt{n+1}} \to 0,$$

and so the series converges.
(f) We use the limit comparison test with $b_n = 1/n$. As

$$\frac{a_n}{1/n} = \frac{n^2-n}{n^2+1} \to 1,$$

and $\sum_n b_n$ is the harmonic series, so it diverges. The series diverges.
(g) We rewrite, using algebra,

$$a_n = \frac{-1}{n(n+1)} = -\frac{1}{n} + \frac{1}{n+1}.$$

This is a telescoping series that converges to -1.

Solution to 3(b). False. The idea is that if $\limsup(na_n) = 2$, most terms may vanish. For example, assume that $a_n = 0$ expect when n is a perfect square, when it is $2/n$. The sequence na_n may be divided into two subsequences, one constantly 0, and one constantly 2. So, $\limsup(na_n) = 2$. Moreover,

$$\sum_{n=1}^{\infty} a_n = \sum_{k=1}^{\infty} \frac{2}{k^2},$$

which is a convergent series.

Solution to 4. (a) We can rewrite

$$G = \bigcup_{a\in A} V_r(a) = \bigcup_{a\in A} (a-r, a+r),$$

which is open as a union of open sets, namely open intervals.
(b) Not necessarily. Note first that

$$B = \bigcup_{a\in A} [a-r, a+r].$$

Take $A = \{1/n : n \in \mathbb{N}\}$ and $r = 1$. Then

$$B = \bigcup_{n\in\mathbb{N}} [-1+1/n, 1+1/n] = [0,2] \cup [-1/2, 3/2] \cup [-2/3, 4/3] \cup \ldots = (-1, 2],$$

which is not closed, as it does not include its accumulation point -1.
(c) Assume $x_n \in B = \cup_{a\in A}[a-r, a+r]$, and $x_n \to x$. So, there are $a_n \in A$ so that $|x_n - a_n| \leq r$. Then $|a_n| \leq |x_n| + |a_n - x_n| \leq |x_n| + r$. As (x_n) is a bounded sequence, so is (a_n). By Bolzano-Weierstrass, there is a convergent subsequence (a_{n_k}), with $a_{n_k} \to a$. As A is closed, $a \in A$. But then $|x_{n_k} - a_{n_k}| \to |x - a|$ and, as $|x_{n_k} - a_{n_k}| \leq r$, $|x - a| \leq r$ by the order theorem. It follows that $x \in [a-r, a+r] \subseteq B$.
(d) If A is compact, then A is also closed, so we know that B is closed, from (c). So we only need to prove that B is bounded. As A is bounded, there exists an M so that $|a| \leq M$ for all $a \in A$. If $b \in B$, there is an $a \in A$ so that $b \in [a-r, a+r]$, i.e., $|b - a| \leq r$. Then $|b| \leq |a| + |b-a| \leq M + r$, thus B is bounded.

Solution to 5. (a) False. For example, $A = [0, 1]$ is connected, but $A^c = (-\infty, 0) \cup (1, \infty)$ is not connected.
(b) False. For example, $A = [0, \infty)$ is connected, and so is $A^c = (-\infty, 0)$.
(c) True. As intersection of two closed sets, $A \subseteq \mathbb{R}$ is closed. As $A \cap [0, 1] \subseteq [0, 1]$, and $[0, 1]$ is bounded, $A \cap [0, 1]$ is bounded. A closed bounded set is compact.
(d) False. The accumulation point 0 is not in the set
(e) True. There exists an M so that $|a| \leq M$ for all $a \in A$, that is, $A \subseteq [-M, M]$. As $[-M, M]$ is closed, $\overline{A} \subset [-M, M]$, and so $\overline{A}$ is included in a bounded interval, thus bounded.
(f) False. The $A = \mathbb{N}$ is unbounded, but its interior is empty.