Math 25, Fall 2014.
Dec. 16, 2014.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ______________________________

ID#: ______________________

Instructions: Each of the 8 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 10 pages (including this one) with 8 problems.

1	
2	
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8	
TOTAL	

1. Assume $x_1 = a$ and

$$x_{n+1} = \frac{2}{5 - 2x_n} \qquad \text{for } n \in \mathbb{N}.$$

(a) Assume $a = 1$. Show that $0 \le x_n \le 2$ for all $n \in \mathbb{N}$, and that the sequence is decreasing. Then show that $\lim_{n\to\infty} x_n$ exists and compute the limit.

To prove $0 \le x_n \le 2$ for all $n \in \mathbb{N}$:

$(n=1)$ $x_1 = 1 \in [0,2]$.

$(n \to n+1)$ Assume $x_n \in [0,2]$, Then $5-2x_n \in [1,5]$

and $\frac{2}{5-2x_n} \in [\frac{2}{5}, 2] \subseteq [0,2]$.

To prove that $x_{n+1} < x_n$ for all $n \in \mathbb{N}$:

$(n=1)$ $x_2 = \frac{2}{3} < 1 = x_1$.

$(n \to n+1)$ If $x_{n+1} < x_n$, then $5 - 2x_{n+1} > 5 - 2x_n$

and (as we know that both sides are positive, as $x_n, x_{n+1} \le 2$)

$\frac{2}{5-2x_{n+1}} < \frac{2}{5-2x_n}$, that is, $x_{n+2} < x_{n+1}$.

Therefore, x_n is a decreasing bounded sequence,

thus convergent. Let $x = \lim x_n$. Then

$$x = \frac{2}{5-2x}, \quad 5x - 2x^2 = 2,$$

$$2x^2 - 5x + 2 = 0 \qquad x = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4}$$

$x = 1/2$ or 2. But $x_1 < 2$ and

the sequence decreases, so $x < 2$.

Answer. $\lim x_n = x = \underline{\underline{1/2}}$.

Problem 1, continued.
(b) Now assume $a = 0$. Show that the sequence is increasing and that it converges to the same limit as in (a).

The proof that $0 \le x_n \le 2$ for all n from (a) is still valid. To prove by induction that $x_{n+1} > x_n$ for all n by induction:

$(n=1)$ $x_2 = \frac{2}{5} > x_1$.

$(n \to n+1)$ $x_{n+1} > x_n \Rightarrow 5 - 2x_{n+1} < 5 - 2x_n$.

both > 0

$$\Rightarrow \frac{1}{5-2x_{n+1}} > \frac{1}{5-2x_n} \Rightarrow x_{n+2} > x_{n+1}$$

Also, $x_n \le \frac{1}{2}$:

$(n=1)$ $x_1 = 0 \le \frac{1}{2}$.

$(n \to n+1)$ If $x_n \le \frac{1}{2}$, then $5 - 2x_n \ge 4$

and $\frac{2}{5-2x_n} \le \frac{1}{2}$, so $x_n \le 1/2$.

The sequence is increasing an bounded by $1/2$, so $x = \lim x_n$ exists and $x \le 1/2$.

Also, x must satisfy the same equation as in (a). So $\underline{\underline{x = 1/2}}$.

2. (a) Assume $A \subseteq \mathbb{R}$. Define precisely what this statement means: A is an open set.

For every $x \in A$, there exist an $\varepsilon > 0$ so that $(x-\varepsilon, x+\varepsilon) \subseteq A$.

(b) Is the set $[0, 1)$ open?

No, $0 \in [0,1)$, but for every $\varepsilon > 0$, $(-\varepsilon, \varepsilon) \not\subseteq [0,1)$.

(c) Is the set $\cup_{n=1}^{\infty}(n, n + 1/n)$ open?

Yes. Any union of open sets (open intervals in this case) is open.

(d) Determine the boundary and the interior of the set $[0, 2] \setminus \{1\}$.

$[0,2] \setminus \{1\} = [0,1) \cup (1,2]$

Interior: $(0,1) \cup (1,2)$

Boundary: $\{0,1,2\}$

3. (a) Assume $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Define precisely what these two statements mean: x is a limit point of A; A is a closed set.

x is a limit point: for every $\varepsilon > 0$,
$A \cap (V_\varepsilon(x) \setminus \{x\}) \neq \emptyset$.

A is closed: A contains all its limit points.

(We denote $V_\varepsilon(x) = (x-\varepsilon, x+\varepsilon)$.)

(b) Assume A is bounded above. Assume that $\sup A \notin A$. Show that $\sup A$ is a limit point of A.

Let $s = \sup A$ and $\varepsilon > 0$. Then there exists an $a \in A$ so that $a > a - \varepsilon$. As s is an upper bd. for A but $s \notin A$, $a < s$. Therefore $a \in (s-\varepsilon, s)$ and

$$a \in (V_\varepsilon(s) \setminus \{s\}) \cap A \neq \emptyset.$$

(c) Assume A is bounded above and closed. Prove that $\sup A \in A$. Assume $\sup A \notin A$. We proved in (b) that $\sup A$ is then a limit pt. of A, so A does not contain one of its limit pts, thus A is not closed.

(d) Assume A is bounded above and open. Prove that $\sup A \notin A$. Let $s = \sup A$.

If $s \in A$, and A is open, there is an $\varepsilon > 0$ so that $(s-\varepsilon, s+\varepsilon) \subseteq A$. But this means, for example that

$$s < s + \varepsilon/2 \in A$$

and s is not an upper bound for A, contradiction.

4. (a) Assume $A \subseteq \mathbb{R}$. Define precisely what this statement means: A is compact.

Every sequence (x_n) with $x_n \in A$ has
a ~~[illegible]~~ convergent subsequence (x_{n_k}) with $\lim_k x_{n_k} \in A$.

(b) Is the set $\{\frac{n}{n+7} : n \in \mathbb{N}\}$ compact?

No, $x_n = \dfrac{n}{n+7}$ of elements in this set converges
to 0 which is not in the set. Any subsequence
x_{n_k} then also converges to 0.

(c) Is the set $[0, 1] \cup \{2\}$ compact?

Yes. By Heine-Borel Theorem, any closed and
bounded set is compact. This set is obviously bounded,
and is closed as a union of two closed sets $[0,1]$ and $\{2\}$.

(d) True or false: If A^c is compact, then A is open and unbounded.

True. A^c is compact $\Rightarrow A^c$ closed $\Rightarrow A$ open.
Also, A cannot be bounded: A^c is bounded (as
a compact set), so if A were also bounded, then
$A \cup A^c = \mathbb{R}$ would also be bounded, contradiction.

(e) Assume A is compact and $\epsilon > 0$. Prove that there exist finitely many elements $x_1, \ldots, x_n \in A$ so that $A \subset \cup_{i=1}^{n}(x_i - \epsilon, x_i + \epsilon)$.

Take the open cover $\mathcal{F} = \{(x-\varepsilon, x+\varepsilon) ; x \in A\}$.
Then there exists a finite subcover
$\{(x_1 - \varepsilon, x_1+\varepsilon), \ldots, (x_n - \varepsilon, x_n + \varepsilon)\}$,
for some $x_1, \ldots, x_n \in A$. This means

$$A \subseteq \bigcup_{i=1}^{n} (x_i - \varepsilon, x_i + \varepsilon)$$

5. For each of the following series, determine (with proof) whether it converges absolutely, converges conditionally, or diverges:

(a) $\underbrace{\sum_{k=1}^{\infty} \left(\frac{7k+3}{8k+5}\right)^{7k}}_{a_k}$ Converges absolutely:

Root test

$$\lim_{k\to\infty} a_k^{1/k} = \left(\frac{7k+3}{8k+5}\right)^7 = \left(\frac{7}{8}\right)^7 < 1.$$

(b) $\sum_{k=1}^{\infty} \underbrace{(-1)^k \frac{\sqrt{k}}{k^2+1}}_{a_k}$ Converges absolutely

$$|a_k| = \frac{\sqrt{k}}{k^2+1} \leq \frac{1}{k^{3/2}}$$

and $\sum_1 \frac{1}{k^{3/2}}$ converges as $3/2 > 1$.

(c) $\sum_{k=1}^{\infty} \underbrace{(-1)^k \frac{1}{3k-2}}_{a_k}$ Converges conditionally.

As $\frac{1}{3k-2}$ decreases to 0, the series converges by the alternating series test. However, $|a_k| = \frac{1}{3k-2}$ and $\lim_{k\to\infty} \frac{|a_k|}{1/k} = \lim_{k\to\infty} \frac{k}{3k-2} = 1/3$ so the series $\sum_1 |a_k|$ diverges by the limit comparison test.

(d) $\sum_{k=1}^{\infty} \left(\frac{1}{k} + \frac{1}{2^k}\right)$

Diverges

$$\frac{1}{k} + \frac{1}{2^k} \geqslant \frac{1}{k} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k} = \infty$$

(Harmonic series)

6. Assume that $a_n > 0$ for all $n \in \mathbb{N}$. For each statement below, prove it or find a counterexample.

(a) If $\sum_{n=1}^{\infty}(a_n - 5)$ converges, then $\lim a_n = 5$.

Yes. $\lim_{n\to\infty}(a_n - 5) = 0$, so $\lim_{n\to\infty} a_n = 5$.

(b) If $a_{n+1} < a_n$ for all $n \in \mathbb{N}$, then the sequence (a_n) is Cauchy.

Yes. Such sequence is bounded (all terms are in $[0, a_1]$) and monotone, thus convergent, thus Cauchy.

(c) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

No. Take $a_n = \frac{1}{n^2}$.

(d) If $\limsup n^2 a_n = 1$, $\sum_{n=1}^{\infty} a_n$ converges.

Yes. Except for finitely many n, $n^2 a_n \le 2$, thus: $a_n \le \frac{2}{n^2}$. The series converges by comparison with $\sum \frac{2}{n^2}$, a convergent series.

7. (a) Assume (a_n) is a sequence of real numbers and $a \in \mathbb{R}$. Define precisely what this statement means: $\lim a_n = a$.

For every $\varepsilon > 0$, there is an $N \in \mathbb{N}$, so that $|a_n - a| < \varepsilon$ for $n \geq N$.

(b) Assume $\lim a_n = a$. Explain why this statement is false: There exists an $\epsilon > 0$ so that there are infinitely many terms of (a_n) outside $(a - \epsilon, a + \epsilon)$.

This statement is false because it is the exact negation of the one in (a).

(c) Let $a_n = \dfrac{n^2 + n}{n^2 + 1}$. Compute $a = \lim a_n$. (You may use algebraic and order limit theorems, but give full justification.)

$$a_n = \frac{1 + \frac{1}{n}}{1 + \frac{1}{n^2}}$$

As $\frac{1}{n} \to 0$ and $\frac{1}{n^2} \to 0$, $\lim a_n = 1$.

(d) Let $a_n = \dfrac{3^n + 2^n}{3^{n+1} + 2^{n+1}}$. Compute $a = \lim a_n$. (Again, you may use algebraic and order limit theorems.)

$$a_n = \frac{1 + \left(\frac{2}{3}\right)^n}{3 + 2\left(\frac{2}{3}\right)^n}$$

as $\left(\frac{2}{3}\right)^n \to 0$ $\left(\frac{2}{3} < 1\right)$, $\lim a_n = \frac{1}{3}$.

8. Assume (a_n) is a sequence of real numbers.
(a) True or false: If the sequence is bounded, then it has a convergent subsequence.

True. This is Bolzano-Weierstrass thm.

(b) True or false: If the sequence is unbounded, then it has no convergent subsequence.

Not true. Take the sequence (a_n) given by $a_n = \begin{cases} 0 & n \text{ even} \\ n & n \text{ odd} \end{cases}$, Then (a_n) is unbounded by $a_{2k} = 0$ is the convergent subsequence.

(c) Prove: If the sequence is unbounded, and $a_n \geq 0$ for all $n \in \mathbb{N}$, then there exists a subsequence that diverges to ∞.

Because (a_n) is unbounded, for any $k \in \mathbb{N}$, we can choose n_k so that $a_{n_k} \geq k$ and so that $n_k > n_{k-1}$.

Then $\lim a_{n_k} \geq \lim k = \infty$.

(d) Assume that $\lim(4a_n - a_n^2) = 3$, and that $\lim a_n$ does not exist. Determine $\limsup a_n$ and $\liminf a_n$.

As $(4a_n - a_n^2) = a_n(4 - a_n) \geq 0$ except for finitely many n, $0 \leq a_n \leq 4$ except for finitely many n (if either $a_n < 0$ or $a_n > 4$, $a_n(4-a_n) < 0$). Thus any convergent subsequence has finite limit. If x is any subsequential limit, then $4x - x^2 = 3$, $x^2 - 4x + 3 = 0$, $(x-3)(x-1) = 0$ $x = 1$ or $x = 3$. Since $\lim a_n$ does not exist, both 1 and 3 must be subsequential limits. So $\limsup a_n = 3$ and $\liminf a_n = 1$.