Math 127A, Fall 2019.

Practice Final Exam

Note. These problems are longer than the actual exam. My advice is to give yourself 120 minutes and no distractions to write out the solutions. The first 8 problems are from the sample final and the other 3 are from Discussion Problems 10, where the respective solutions are provided.

1. Assume $x_1 = a$ and

$$x_{n+1} = \frac{2}{5 - 2x_n} \qquad \text{for } n \in \mathbb{N}.$$

(a) Assume $a = 1$. Show that $0 \leq x_n \leq 2$ for all $n \in \mathbb{N}$, and that the sequence is decreasing. Then show that $\lim_{n\to\infty} x_n$ exists and compute the limit.
(b) Now assume $a = 0$. Show that the sequence is increasing and that it converges to the same limit as in (a).

2. (a) Assume $A \subseteq \mathbb{R}$. Define precisely what this statement means: A is an open set.
(b) Is the set $[0, 1)$ open?
(c) Is the set $\cup_{n=1}^{\infty}(n, n + 1/n)$ open?
(d) Determine the boundary and the interior of the set $[0, 2] \setminus \{1\}$.

3. (a) Assume $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Define precisely what these two statements mean: x is a limit point of A; A is a closed set.
(b) Assume A is bounded above. Assume that $\sup A \notin A$. Show that $\sup A$ is a limit point of A.
(c) Assume A is bounded above and closed. Prove that $\sup A \in A$.
(d) Assume A is bounded above and open. Prove that $\sup A \notin A$.

4. (a) Assume $A \subseteq \mathbb{R}$. Define precisely what this statement means: A is compact.
(b) Is the set $\{\frac{n}{n+7} : n \in \mathbb{N}\}$ compact?
(c) Is the set $[0, 1] \cup \{2\}$ compact?
(d) True or false: If A^c is compact, then A is open and unbounded.
(e) Assume A is compact and $\epsilon > 0$. Prove that there exist finitely many elements $x_1, \ldots, x_n \in A$ so that $A \subset \cup_{i=1}^{n}(x_i - \epsilon, x_i + \epsilon)$.

5. For each of the following series, determine (with proof) whether it converges absolutely, converges conditionally, or diverges:

(a) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{7k+3}{8k+5}\right)^{7k}$

(b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k}}{k^2+1}$

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{1}{3k-2}$

(d) $\displaystyle\sum_{k=1}^{\infty} \left(\frac{1}{k} + \frac{1}{2^k}\right)$

6. Assume that $a_n > 0$ for all $n \in \mathbb{N}$. For each statement below, prove it or find a counterexample.

(a) If $\sum_{n=1}^{\infty}(a_n - 5)$ converges, then $\lim a_n = 5$.

(b) If $a_{n+1} < a_n$ for all $n \in \mathbb{N}$, then the sequence (a_n) is Cauchy.

(c) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges.

(d) If $\limsup n^2 a_n = 1$, $\sum_{n=1}^{\infty} a_n$ converges.

7. (a) Assume (a_n) is a sequence of real numbers and $a \in \mathbb{R}$. Define precisely what this statement means: $\lim a_n = a$.

(b) Assume $\lim a_n = a$. Explain why this statement is false: There exists an $\epsilon > 0$ so that there are infinitely many terms of (a_n) outside $(a - \epsilon, a + \epsilon)$.

(c) Let $a_n = \dfrac{n^2 + n}{n^2 + 1}$. Compute $a = \lim a_n$. (You may use algebraic and order limit theorems, but give full justification.)

(d) Let $a_n = \dfrac{3^n + 2^n}{3^{n+1} + 2^{n+1}}$. Compute $a = \lim a_n$. (Again, you may use algebraic and order limit theorems.)

8. Assume (a_n) is a sequence of real numbers.

(a) True or false: If the sequence is bounded, then it has a convergent subsequence.

(b) True or false: If the sequence is unbounded, then it has no convergent subsequence.

(c) Prove: If the sequence is unbounded, and $a_n \geq 0$ for all $n \in \mathbb{N}$, then there exists a subsequence that diverges to ∞.

(d) Assume that $\lim(4a_n - a_n^2) = 3$, and that $\lim a_n$ does not exist. Determine $\limsup a_n$ and $\liminf a_n$.

9. Assume $f : \mathbb{R} \to \mathbb{R}$ is a function.

(a) Assume that f is continuous and $K \subseteq \mathbb{R}$ is compact. Is it necessarily true that $f(K)$ is compact?

(b) Assume that f is continuous and $K \subseteq \mathbb{R}$ is compact. Is it necessarily true that $f^{-1}(K)$ is compact?

(c) Assume that f is continuous and $A \subseteq \mathbb{R}$ is bounded. Is it necessarily true that $f(A)$ is bounded?

(d) Assume that $f(K)$ is compact for every $K \subseteq \mathbb{R}$. Is f necessarily continuous?

(e) Assume that f is continuous. Show that its set of zeros, $\{x \in \mathbb{R} : f(x) = 0\}$ is closed.

10. Prove that $f(x) = \sqrt{x^2 + 1}$ is uniformly continuous on $(0, \infty)$.

11. (a) Let f be a continuous function on the closed interval $[0, 1]$ with range also contained in $[0, 1]$. Prove that f must have a fixed point; that is, show that there exists an $x \in [0, 1]$ so that $f(x) = x$. (b) Is the conclusion in (a) true if f is a continuous function on the *open* interval $(0, 1)$ with range also contained in $(0, 1)$? (c) Is the conclusion in (a) true if f is a continuous decreasing function on the *open* interval $(0, 1)$ with range also contained in $(0, 1)$?