Math 127A, Fall 2019.
Dec. 11, 2019.

FINAL EXAM

NAME(print in CAPITAL letters, *first name first*): KEY

NAME(sign): ____________________

ID#: ____________________

Instructions: Each of the 8 problems has equal worth. Read each question carefully and answer it in the space provided. *You must show all your work for full credit. Carefully prove each assertion you make unless explicitly instructed otherwise.* Clarity of your solutions may be a factor when determining credit. Calculators, books or notes are not allowed. The proctor has been directed not to answer any interpretation questions.

Make sure that you have a total of 9 pages (including this one) with 8 problems.

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8	
TOTAL	

1. Assume $a_1 = a$ and

$$a_{n+1} = \sqrt{4 + 3a_n} \qquad \text{for } n \in \mathbb{N}.$$

Assume $a = 0$. Show that $0 \leq a_n \leq 4$ for all $n \in \mathbb{N}$, and that the sequence is increasing. Then show that $\lim_{n\to\infty} a_n$ exists and compute the limit.

To show that $0 \leq a_n \leq 4$ for $n \geq 1$:

$(n=1)$ $\quad a_n = 0 \in [0,4]$

$(n \to n+1)$ $\quad$ If $0 \leq a_n \leq 4$, then $4 \leq 4 + 3a_{n+1} \leq 16$,

so $a_{n+1} = \sqrt{4+3a_n} \in [2,4] \subseteq [0,4]$.

To show that $a_n \leq a_{n+1}$ for $n \geq 1$:

$(n=1)$ $\quad 0 = a_1 \leq a_2 = \sqrt{4} = 2$.

$(n \to n+1)$ $\quad$ If $a_n \leq a_{n+1}$, then $4 + 3a_n \leq 4 + 3a_{n+1}$

and so $\sqrt{4+3a_n} \leq \sqrt{4+3a_{n+1}}$, i.e., $a_{n+1} \leq a_{n+2}$.

As (a_n) is a bounded increasing sequence, $\lim a_n = a$ exists. By algebraic theorems:

$$a = \sqrt{4+3a}$$

$$a^2 = 4 + 3a$$

$$a^2 - 3a - 4 = 0$$

$$(a+1)(a-4) = 0$$

~~therefore~~ As $a \geq 0$ (because all $a_n \geq 0$),

$\underline{a = 4}$,

2. (a) Assume $A \subseteq \mathbb{R}$. Define precisely what this statement means: A is an open set.

$$(\forall x \in A)\,(\exists \varepsilon > 0)\,\big((x-\varepsilon, x+\varepsilon) \subseteq A\big)$$

(b) Is the set $(0,1) \cup \{2\}$ open?

No. $(\forall \varepsilon > 0)\,\big((2-\varepsilon, 2+\varepsilon)$ includes a pt. outside A, say the point $2 + \varepsilon/2$.

(c) Determine the closure of the set in (b).

The limit pts. are 0 and 1, so the closure is

$$[0,1] \cup \{2\}$$

(d) Determine the boundary and the interior of the set $\underbrace{([0,1] \cap \mathbb{Q}) \cup [2,3]}_{A}$.

$\partial A = [0,1] \cup \{2,3\}$, as every $x \in [0,1]$ is a limit of a sequence of rational (resp. irrational) pts. in $[0,1]$.

$A^\circ = (2,3)$, as no pt. in ∂A can be in A°.

3. (a) Assume $A \subseteq \mathbb{R}$ and $s \in \mathbb{R}$. Define precisely what this statement means: $s = \sup A$.

s is an upper bd.: $(\forall x \in A)\,(x \leq s)$

s is the least upper bd.: $(\forall \varepsilon > 0)\,(\exists\, x \in A)$
$(x > s - \varepsilon)$

(b) Assume $K \subseteq \mathbb{R}$. Define precisely what this statement means: K is compact.

For every sequence (x_n), $x_n \in K$, there is a subsequence x_{n_k} so that $\lim_{k \to \infty} x_{n_k}$ exists and is in K.

(c) True or false: if $K \subseteq \mathbb{R}$ is a compact, then $\sup K \in K$.

True. We proved this in class.

(d) Let $A = \{\frac{2n}{n+1} : n \in \mathbb{N}\}$. Determine $\sup A$ and determine whether A is compact.

$\frac{2n}{n+1} = \frac{2}{1+\frac{1}{n}}$ is an increasing sequence with limit 2. So $\sup A = 2$. As $\sup A \notin A$ $\left(\frac{2}{1+\frac{1}{n}} < 2 \text{ for all } n \in \mathbb{N}\right)$, A is not compact.

4. For each of the following series, determine (with proof) whether it converges absolutely, converges conditionally, or diverges:

(a) $\sum_{k=1}^{\infty} \underbrace{\boxed{\left(\frac{5k+7}{7k+5}\right)^k}}_{a_k}$

Root test

$$a_k^{1/k} = \frac{5k+7}{7k+5} \to \frac{5}{7} < 1$$

Converges absolutely.

(b) $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{\sqrt{7k+5}}$

$a_k = \frac{1}{\sqrt{7k+5}}$ is a

decreasing sequence that converges to 0,

so the series converges by alternating series test.

Also $\sum \frac{1}{\sqrt{7k+5}}$ diverges by limit comparison with

$b_k = \sum \frac{1}{\sqrt{k}}$: $\frac{a_k}{b_k} = \frac{\sqrt{k}}{\sqrt{7k+5}} = \frac{1}{\sqrt{7+\frac{5}{k}}} \to \frac{1}{\sqrt{7}}$,

Converges conditionally

(c) $\sum_{k=1}^{\infty} \underbrace{\frac{\sqrt{5k+7}}{7k^2+5}}_{a_k}$

Limit comparison with $\sum b_k$, $b_k = \frac{1}{k^{3/2}}$, which converges; (p-series, $p = 3/2 > 1$)

$$\frac{a_k}{b_k} = \frac{\sqrt{5k+7}\cdot k^{3/2}}{7k^2+5} = \frac{\sqrt{5k^4+7k^3}}{7k^2+5} = \frac{\sqrt{5+\frac{7}{k}}}{7+5/k^2} \to \frac{\sqrt{5}}{7}$$

Converges absolutely

(d) $\sum_{k=1}^{\infty}(-1)^k\frac{\sqrt{5k+7}}{7k^2+5}$

Converges absolutely, by (c)

5. Assume that $a_n > 0$ for all $n \in \mathbb{N}$. For each statement below, prove it or find a counterexample.

(a) If $\lim a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

No. Counterexample is the harmonic series with $a_n = \frac{1}{n}$.

(b) If $a_n < 1/2^n$ for all $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} a_n$ converges.

Yes. Comparison with the geometric series: $\sum_{n=1}^{\infty} 1/2^n = 1$.

(c) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{2a_n}{2+a_n}$ converges.

Yes. Limit comparison test and n-term test (as $\sum a_n$ converges, $a_n \to 0$):

$$\lim_{n\to\infty} \frac{\frac{2a_n}{1+a_n}}{a_n} = \frac{2}{1+a_n} = 2.$$

(d) If $\limsup n^{3/2} a_n = 2$, then $\sum_{n=1}^{\infty} a_n$ converges.

Yes. If $\limsup n^{3/2} a_n = 2$, then $\exists N \in \mathbb{N}$ so that $n^{3/2} a_n \leq 3$ for $n \geq N$.

So, for $n \geq N$, so that $a_n \leq \frac{3}{n^{3/2}}$, and $\sum \frac{1}{n^{3/2}}$ converges (as it is p-series with $p = \frac{3}{2} > 1$)

6. (a) Assume (a_n) is a sequence of real numbers and $a \in \mathbb{R}$. Define precisely what this statement means: $\lim a_n = a$.

$(\forall \varepsilon > 0)\ (\exists N \in \mathbb{N})\ (n \geq N \Rightarrow |a_n - a| < \varepsilon)$

(b) True or false: if $0 \leq a_n \leq 4$, then (a_n) has a convergent subsequence.

Yes. This follows from the Bolzano-Weierstrass theorem.

(c) Assume that (a_n) is decreasing and $\lim a_n = 4$. Determine $\bigcap_{n \in \mathbb{N}} (0, a_n]$. $= \underline{\underline{(0,4]}}$

We know that $4 = \inf\{a_n : n \in \mathbb{N}\}$, so $(0,4] \subseteq (0,a_n]$ for all n. Also, for any $x > 4$, $\exists n$ so that $a_n < x$, so that $x \notin (0, a_n]$.

(d) Assume that (a_n) is bounded and $\lim(a_{2n} - 2a_n) = 0$. Show that (a_n) is convergent and determine $\lim a_n$.

Consider $a = \limsup a_n \in \mathbb{R}$. Then there exists a subsequence $a_{n_k} \to a$. But then another subsequence $a_{2n_k} = (a_{2n_k} - 2a_{n_k}) + 2a_{n_k} \to 0 + 2a = 2a$.

So $2a \leq a$ (as a is the largest subsequential limit) and so $a \leq 0$. If $b = \liminf a_n \in \mathbb{R}$, then $a_{n_k} \to b$ for some subsequence. Then (as above) $a_{2n_k} \to 2b$ and so $2b \geq b$ (as b is the smallest subsequential limit) and so $b \geq 0$.

So we have $0 \leq b \leq a \leq 0$, and $b = a = 0$.

Conclusion: $\lim a_n = 0$.

7. (a) Assume that $f : \mathbb{R} \to \mathbb{R}$ is a function. Define precisely what this statement means: f is continuous on $\mathbb{R}$.

$$(\forall x \in \mathbb{R})(\forall \varepsilon > 0)(\exists \delta > 0)(\forall y \in \mathbb{R})(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon)$$

(b) Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(0) = 7$. Assume that $x_n \in \mathbb{R}$ and $\lim x_n = 0$. Compute $\lim f(x_n)$.

For a cont. function f, $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Therefore, $\lim f(x_n) = f(0) = 7$.

(c) Assume $f : [0, \infty) \to \mathbb{R}$ is given by

$$\begin{cases} \frac{\sqrt{x}-1}{x-1} & x \neq 1 \\ c & x = 1 \end{cases}$$

For which value of $c \in \mathbb{R}$ is f is continuous on its domain $[0, \infty)$?

For any value of c, f is continuous on $[0, \infty) \setminus \{1\}$.

$$\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

$\boxed{c = 1/2}$ so that $\lim_{x \to 1} f(x) = f(1)$, and so f is cont. at 1.

(d) True or false: If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then its range $f(\mathbb{R})$ is an open set.

No, Example: $f(x) = \sin x$, $f(\mathbb{R}) = [0, 1]$

(or $f(x) = |x|$, $f(\mathbb{R}) = [0, \infty)$)

8. Assume, in all parts of this problem, that $f : [0,1] \to \mathbb{R}$ is continuous and that $f([0,1]) \subseteq [-1,2]$.
(a) Show that there exists an $x \in [0,1]$ so that $f(x) = \sqrt{8x^2+x} - 1$.

$g(x) = f(x) - (\sqrt{8x^2+x} - 1)$, which is continuous on $[0,1]$

$g(0) = f(0) + 1 \geq 0$

$g(1) = f(1) - 2 \leq 0$

By IVT, $\exists x \in [0,1]$ so that $f(x) = 0$,

(b) True or false: The function f is uniformly continuous on its domain $[0,1]$.

True. Any continuous function on a compact set is uniformly continuous, and $[0,1]$ is compact.

(c) True or false: The set $\left\{ \frac{1}{3-f(x)} : x \in [0,1] \right\}$ is compact. True.

Let $g(x) = \frac{1}{3-x}$. Then g is continuous on $[-1,2]$, so on $f([0,1])$. By the composition theorem, $g \circ f$ is continuous on $[0,1]$, and so its range $g \circ f([0,1])$ is compact. As $g \circ f(x) = \frac{1}{3-f(x)}$, $g \circ f([0,1])$ is exactly the above set.

(d) True or false: There exists an $x \in [0,1]$ so that $f(x) = 2x$.

False. The constant function $f(x) = -1$ is an example: $2x \geq 0 > -1$ for all $x \in [0,1]$.