Math 127A, Fall 2019.

HW 1 Solutions (mostly adapted from Abbott's Instructor's Manual)

1.2.1. (a) We prove this by contradiction. Assume that there exist integers p and q satisfying

$$\left(\frac{p}{q}\right)^2 = 3.$$

We may assume that p and q have no common factor. From the above equation we get

$$p^2 = 3q^2.$$

Therefore $3|p^2$ and hence $3|p$. This allows us to write $p = 3k$ where k is an integer. After substituting $3k$ for p, we get $(3k)^2 = 3q^2$, which can be simplified to $3k^2 = q^2$. This implies $3|q^2$, then $3|q$. Thus we have shown p and q have a common factor, namely 3. But they were originally assumed to have no common factor, a contradiction.

A similar argument will work for 6 as well because we get $p^2 = 6q^2$ which implies p is a multiple of both 2 and 3, hence a multiple of 6. After the argument as above, we conclude q is also a multiple of 6. Therefore, 6 must be irrational.
(b) In this case, the fact that $4|p^2$ does not imply $4|p$. Thus, the above proof breaks down at this point.

1.2.3. (a) False, as seen in Example 1.2.2.
(c) False. Consider sets $A = \{1, 2, 3\}$, $B = \{3, 6, 7\}$, and $C = \{5\}$. Note that $A \cap (B \cup C) = \{3\}$ is not equal to $(A \cap B) \cup C = \{3, 5\}$. (It is however true that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \subset (A \cap B) \cup C$.
(d) and (e) are both true.

1.2.6. (d) First note that $|a| = |a - b + b| \leq |a - b| + |b|$. Taking $|b|$ to the left side of the inequality we get $|a| - |b| \leq |a - b|$. Reversing the roles of a and b in the previous argument gives $|b| - |a| \leq |b - a|$, thus (after multiplying by -1), $-|b - a| \leq |a| - |b|$ Because $|a - b| = |b - a|$, the result follows.

1.2.7. (c) We have to show that $y \in g(A \cap B)$ implies $y \in g(A) \cap g(B)$. If $y \in g(A \cap B)$, then there exists an $x \in A \cap B$ with $g(x) = y$. But this implies that $x \in A$ and $x \in B$ and hence $g(x) \in g(A)$ and $g(x) \in g(B)$. Therefore, $g(x) = y \in g(A) \cap g(B)$.

1.2.9. (b) We verify this by definition:

$$\begin{aligned}
&x \in g^{-1}(A) \cap g^{-1}(B) \\
\iff &x \in g^{-1}(A) \text{ and } x \in g^{-1}(B) \\
\iff &g(x) \in A \text{ and } g(x) \in B \\
\iff &g(x) \in A \cap B \\
\iff &x \in g^{-1}(A \cap B).
\end{aligned}$$

Similarly,

$$\begin{aligned} & x \in g^{-1}(A) \cup g^{-1}(B) \\ \iff & x \in g^{-1}(A) \text{ or } x \in g^{-1}(B) \\ \iff & g(x) \in A \text{ or } g(x) \in B \\ \iff & g(x) \in A \cup B \\ \iff & x \in g^{-1}(A \cup B). \end{aligned}$$

1.2.11. (a) There exist two real numbers a and b satisfying $a < b$, such that for all $n \in \mathbb{N}$ we have $a + 1/n \geq b$. (The original claim is true.)
(b) For any real number $x > 0$, there exists an $n \in \mathbb{N}$ such that $1/n \leq x$. (The negation is true.)
(c) There exist two distinct distinct real numbers with no rational number between them. (The original claim is true.)

1.2.12. (a) For $n = 1$, we check: $y_1 = 6 > -6$. For the induction step, we want to show that if we have $y_n > -6$, then it follows that $y_{n+1} > -6$. Starting from the induction hypothesis $y_n > -6$, we get $y_{n+1} = (2y_n - 6)/3 > (2(-6) - 6)/3 = -6$, which is the the desired conclusion.
(b) For $n = 1$, we check $y_1 = 6 > 2 = y_2$, proving the base case. For the induction step, we want to show that if we have $y_n \geq y_{n+1}$, then it follows that $y_{n+1} \geq y_{n+2}$. Starting from the induction hypothesis $y_n \geq y_{n+1}$, we can multiply across the inequality by 2, add (-6), and divide by 3 to get $(2y_n - 6)/3 \geq (2y_{n+1} - 6)/3$ which is the the desired conclusion $y_{n+1} \geq y_{n+2}$. By induction, the claim is proved for all $n \in \mathbb{N}$.

1.2.13. (a) The check for $n = 2$ is easy and is omitted. For the induction step, we want to show that if we have

$$(A_1 \cup \ldots \cup A_n)^c = A_1^c \cap \ldots \cap A_n^c,$$

then it follows that

$$(A_1 \cup \ldots \cup A_{n+1})^c = A_1^c \cap \ldots \cap A_{n+1}^c.$$

But we have

$$(A_1 \cup \ldots \cup A_{n+1})^c = ((A_1 \cup \ldots \cup A_n) \cup A_{n+1})^c$$

which by $n = 2$ case equals

$$(A_1^c \cap \ldots \cap A_n^c) \cap A_{n+1}^c$$

and we drop the parentheses to get the desired conclusion. By induction, the claim is proved for all $n \in N$.
(b) The point here is to distinguish between asserting that a statement is true for all values of $n \in \mathbb{N}$ and asserting that it is true in the infinite case. Induction cannot be used when we have an infinite number of sets: it is used to prove facts that hold true for each value of $n \in \mathbb{N}$, bur then we cannot just plug in $n = \infty$!
(c) By definition $x \notin \cup_{n=1}^{\infty} A_n$ exactly when the negation of the following statement is true:

$$(\exists n \in \mathbb{N})(x \in A_n).$$

But the negation is equivalent to

$$(\forall n \in \mathbb{N})(x \notin A_n) \iff (\forall n \in \mathbb{N})(x \in A_n^c) \iff x \in \cap_{n=1}^{\infty} A_n^c.$$