Math 127A, Fall 2019.

HW 3 Solutions (mostly adapted from Abbott's Instructor's Manual)

2.2.2. (a) Let $a_n = \frac{2n+1}{5n+4}$. Pick an $\epsilon > 0$. We must produce an $N \in \mathbb{N}$ so that $n \geq N$ implies $|a_n - \frac{2}{5}| < \epsilon$. After a bit of algebra,

$$\left|a_n - \frac{2}{5}\right| = \frac{3}{4n+10}.$$

So $|a_n - \frac{2}{5}| < \epsilon$ as soon as $n > (3/4)\epsilon^{-1} - 10/4$. One can take, say, N to be the smallest integer larger that ϵ^{-1}.

2.2.5. (b) The sequence a_n satisfies $a_n = 1$ for $n \geq 7$, and so the limit is 1. For *any* $\epsilon > 0$ we can take $N = 7$ to guarantee $|a_n - 1| < \epsilon$ for $n \geq N$. (Here, N is independent of ϵ, which is almost never true!)

2.2.7. (a) Frequently, yes. (In fact, frequently means that $a_n \in A$ for infinitely many n. This is true for all even n in this case.) Eventually, no. (In fact, eventually means that $a_n \in A$ for all but finitely many n. This is not true in this case, as $a_n \notin A$ for infinitely many n, that is, for all odd n.)
(b) Eventually implies frequently but not the reverse. See the above discussion for (a).

2.3.1(b). We will assume $x > 0$. Write

$$\sqrt{x_n} - \sqrt{x} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}.$$

This implies

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}.$$

As $|x_n - x| \to 0$ and $\sqrt{x}$ is a constant,

$$\frac{|x_n - x|}{\sqrt{x}} \to 0$$

by algebraic limit theorem, and then $|\sqrt{x_n} - \sqrt{x}| \to 0$ by order limit theorems (or more precisely by the squeeze theorem, which is the subject of the next exercise).

2.3.3. Pick an $\epsilon > 0$. We need to show that there exists an N so that $n \geq N$ implies $|y_n - l| < \epsilon$. However, we can find an N_1 so that $n \geq N_1$ implies $|x_n - l| < \epsilon$, that is $l - \epsilon < x_n < l + \epsilon$. Similarly, we can find an N_2 so that $n \geq N_2$ implies $l - \epsilon < z_n < l + \epsilon$. Take N to be $\max\{N_1, N_2\}$, the larger of N_1 and N_2. If $n \geq N$, then $n \geq N_1$ and $n \geq N_2$, so that

$$l - \epsilon \leq x_n \leq y_n \leq z_n \leq l + \epsilon$$

and so $|y_n - l| < \epsilon$.

2.3.4. (a) As $a_n \to 0$, also $a_n^2 \to 0$, and the answer is 1.
(b) As $a_n \neq 0$, $\frac{(a_n+2)^2-4}{a_n} = 4 + a_n$, so the limit is 4.
(c) The sequence equals $\frac{2+3a_n}{1+5a_n}$ so it has limit 2.

2.3.9. (a) We are only assuming that a_n is bounded, so we cannot use the algebraic limit theorem, which assumes that a_n converges.

Pick an $\epsilon > 0$. We need to show that there exists an N so that $n \geq N$ will imply $|a_n b_n| < \epsilon$. As a_n is bounded, there exists an M so that $|a_n| \leq M$ for all n. As $b_n \to 0$, there exists an N so that $n \geq N$ implies $|b_n| < \epsilon/M$. Then $n \geq N$ implies

$$|a_n b_n| = |a_n||b_n| \leq M \cdot \frac{\epsilon}{M} = \epsilon.$$

(b) No. For a counterexample, take a_n to be any non-convergent bounded sequence (e.g., $a_n = (-1)^n$) and $b_n = 1$.
(c) Use (a) and the fact that a convergent sequence is bounded.

2.3.10.(a) No, as $\lim a_n$ and $\lim b_n$ may not exist. For a counterexample, take $a_n = b_n = n$.
(b) Yes, by the triangle inequality, $0 \leq ||b_n| - |b|| \leq |b_n - b|$, and $|b_n - b| \to 0$, so that $||b_n| - |b|| \to 0$ by the squeeze theorem, which implies that $|b_n| \to |b|$.
(c) Yes, by $b_n = a_n + (b_n - a_n)$.

2.3.11. (a) Assume $x_n \to x$. Pick an $\epsilon > 0$. Then there exists an N so that $n \geq N$ implies that $|x_n - x| < \epsilon/2$. Also there exists an M so that $|x_n| \leq M$ for all n (as the sequence x_n is bounded). Then, for $n \geq N$,

$$\begin{aligned}
|y_n - x| &= \frac{1}{n}|(x_1 - x) + \ldots + (x_n - x)| \\
&\leq \frac{1}{n}(|x_1 - x| + \ldots + |x_{N-1} - x| + |x_N - x| \ldots + |x_n - x|) \\
&\leq \frac{1}{n}(|x_1| + |x| + \ldots + |x_{N-1}| + |x| + |x_N - x| \ldots + |x_n - x|) \\
&\leq \frac{1}{n}\left((N-1)(M + |x|) + (n - N + 1)\frac{\epsilon}{2}\right) \\
&\leq \frac{N}{n}(M + |x|) + \frac{\epsilon}{2}
\end{aligned}$$

Now pick an integer N_1 so that $N_1 \geq N$ and $N_1 > \frac{2}{\epsilon}N(M + |x|)$. Then $n \geq N_1$ implies

$$|y_n - x| \leq \frac{N}{N_1}(M + |x|) + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

(b) To get a requested example, let $x_n = (-1)^n$. Then x_n does not converge. However, $x_1 + \ldots + x_n$ is either -1 (odd n) or 0 (even n). It follows that $|y_n| \leq 1/n$ and thus $y_n \to 0$.