Math 127A, Fall 2019.

HW 4 Solutions (mostly adapted from Abbott's Instructor's Manual)

2.4.1. *First proof*. First check with a bit of algebra that for any real number $x < 4$,

$$x \geq 2 - \sqrt{3} \iff \frac{1}{4-x} \geq 2 - \sqrt{3} \qquad \text{and} \qquad x \leq 2 + \sqrt{3} \iff \frac{1}{4-x} \leq 2 + \sqrt{3}.$$

It follows by induction that for every n, $2 - \sqrt{3} < x_n \leq 2 + \sqrt{3}$. Next, we check that $x_{n+1} \leq x_n$, which follows from

$$x_{n+1} \leq x_n \iff 1 + x_n(x_n - 4) \leq 0 \iff (x_n - 2)^2 \leq 3$$

and the last inequaliy says exactly that $|x_n - 2| \leq \sqrt{3}$, which we have already proved. Thus x_n is a decreasing bounded sequence, hence convergent; denote the limit by x. As $x = 1/(4 - x)$ and $x \leq 3$, $x = 2 - \sqrt{3}$.

Second proof. Another way (a bit more straightforward) is to begin by showing that $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$, using induction. This is true for $n = 1$, as $x_1 = 3$, $x_2 = 1$. To do the $(n \to n + 1)$ step, first observe that the induction hypothesis implies that $3 = x_1 \geq x_2 \geq \ldots \geq x_{n+1}$ and so all terms up to $(n + 1)$st one are at most 3. Then the statement for $n + 1$ follows from this chain of implications:

$$x_{n+1} \leq x_n \Longrightarrow 4 - x_{n+1} \geq 4 - x_n \Longrightarrow \frac{1}{4 - x_{n+1}} \leq \frac{1}{4 - x_n} \Longrightarrow x_{n+2} \leq x_{n+1}.$$

So we know that x_n decreases, and so $x_n \leq 3$, and so $x_{n+1} = 1/(4 - x_n) > 0$. So the sequence is nonnegative, thus bounded. As it is also decreasing, it has a limit x, which satisfies $x = 1/(4 - x)$ and $x \leq 3$, so $x = 2 - \sqrt{3}$.

2.4.3(b). This is the sequence defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$. If $0 < x_n < 2$, then $0 < x_{n+1} < 2$, thus $0 < x_n < 2$ for all n, the sequence is bounded. Then $2x_n - x_n^2 = x_n(2 - x_n) > 0$, which implies that $x_n < \sqrt{2x_n} = x_{n+1}$, the sequence is increasing. Thus the limit $x = \lim x_n$ exists, and, as $x_{n+1}^2 = 2x_n$, satisfies $x^2 = 2x$. As $x \neq 0$, $x = 2$.

2.4.5. Observe that $x_n > 0$ for all n (by simple induction). Also, we claim that for any $x > 0$, $x + 2/x \geq 2\sqrt{2}$. Since we cannot use calculus yet, we use algebra to transform the inequality to $x^2 - 2\sqrt{2}x + 2 \geq 0$, which is true as $x^2 - 2\sqrt{2}x + 2 = (x - \sqrt{2})^2$. Thus $x_n \geq \sqrt{2}$ for every n and

$$x_n - x_{n+1} = \frac{1}{2}\left(x_n - \frac{2}{x_n}\right) = \frac{x_n^2 - 2}{2x_n} \geq 0$$

and the sequence decreases. The equation $x = \frac{1}{2}(x + 2/x)$ has the only positive solution $x = \sqrt{2}$. Therefore $\lim x_n = \sqrt{2}$.

The sequence defined by $x_1 = c$, and $x_n = \frac{1}{2}(x_n + c/x_n)$ converges to $\sqrt{c}$, by a very similar proof.

2.4.7. This (and more) was done in lecture and discussion.

2.5.1. (a) Impossible. By Bolzano-Weierstrass, the bounded subsequence must have a convergent subsequence (a sub-subsequence, so to speak), which is also a subsequence of the original sequence.

(b) Let $a_n = 1/(n+1)$ for odd n and $a_n = 1 - 1/n$ for even n.
(c) Let $p_1, p_2, \dots$ be the sequence of primes. Let $a_{p_i^k} = 1/i$ for every k, that is at each power of the i'th prime the element is $1/i$. Set the rest of terms of (a_n) to be 0. (We use prime powers because they are never equal to each other and so ambiguity does not arise.)

2.5.2. (b) True. If (x_n) converges to the limit $x \in \mathbb{R}$, then every subsequence of (x_n) converges to x.
(c) True. If the sequence (x_n) diverges, $\liminf x_n$ and $\limsup x_n$ are different, and they are both subsequential limits.
(d) True. Assume x_n is increasing. If (x_n) diverges, then $\lim x_n = \infty$, and so $\liminf x_n = \infty$, and therefore any subsequence of (x_n) must diverge to ∞.

2.5.5. *First proof*. As $\liminf a_n$ and $\limsup a_n$ are the respectively smallest and largest subsequential limits, the assumption implies that they are both equal to a, Therefore the sequence converges to a.
Second proof. Assume (a_n) does not converge to a. Then there exists an $\epsilon > 0$ so that for every $N \in \mathbb{N}$ there exists an $n \geq N$ so that $|a_n - a| \geq \epsilon$. This is equivalent to saying that there exists a subsequence (a_{n_k}) so that $|a_{n_k} - a| \geq \epsilon$. As (a_n) is bounded, so is (a_{n_k}), and so it has a convergent subsequence $(a_{n_{k_\ell}})$. This is also a subsequence of (a_n) and so $\lim_\ell a_{n_{k_\ell}} = a$. One the other hand, $|a_{n_{k_\ell}} - a| \geq \epsilon$ for every ℓ. This is a contradiction.

2.5.7. Let $a_n = b^n$. Assume first $|b| < 1$. We need to show that $|a_n| \to 0$. However, $|a_n| = |b|^n$ and $0 < |b| < 1$, so $|a_n| \to 0$ by what we proved in class. Now assume $|b| \geq 1$. Then $|a_n| \geq 1$, so that $|a_n|$ cannot converge to 0.