Math 25, Fall 2014.

HW 5 Solutions (mostly adapted from Abbott's Instructor's Manual)

2.6.2. (a) Let $a_n = (-1)^n/n$ (which converges to 0 and is therefore Cauchy, but is not monotone).
(b) Impossible. A Cauchy sequence is bounded and therefore so is every one of its subsequences.
(c) Impossible. Assume that x_n is increasing. If it is divergent, then it diverges to ∞ and then so does any one of its subsequences. So any subsequence diverges, and it cannot be Cauchy, as every Cauchy sequence converges.
(d) Let $a_n = 0$ for even n and $a_n = 1$ for odd n; the even terms form a constant, hence convergent, hence Cauchy, subsequence.

2.6.5. The Cauchy criterion requires that $|s_m - s_n| < \epsilon$ for all $m, n \geq N$ not just when m and n differ by 1. In fact, pseudo-Cauchy requirement is that $\lim_n (s_{n+1} - s_n) = 0$ while the Cauchy requirement is that $\lim_n \sup_{m \geq n} |s_m - s_n| = 0$.
(i) False. Take $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$. Then $s_{n+1} - s_n = \frac{1}{n+1} \to 0$, but s_n diverges to ∞ as the harmonic series diverges.
(ii) True. If $(x_{n+1} - x_n) \to 0$ and $(y_{n+1} - y_n) \to 0$, then $(x_{n+1} + y_{n+1}) - (x_n + y_n) = (x_{n+1} - x_n) + (y_{n+1} - y_n) \to 0$.

2.7.4. (a) Take $x_n = y_n = 1/n$. Then both $\sum_n x_n$ and $\sum_n y_n$ diverge as they are harmonic series, but $\sum_n x_n y_n = \sum_n 1/n^2$ converges.
(b) Take $x_n = (-1)^{n+1}/n$ and $y_n = (-1)^{n+1}$. Then both $\sum_n x_n$ is an alternating harmonic series, thus convergent, and $\sum_n y_n$ is the harmonic series, this divergent.
(c) Impossible, as $y_n = (x_n + y_n) - x_n$ and so $\sum_n y_n$ by the algebraic theorem.
(d) Take $x_n = 0$ for odd n and $x_n = 1/n$ for even n. Then $\sum_n x_n = \sum_k 1/(2k)$, which is one half of the harmonic series, thus diverges.

2.7.7. (a) As $\ell > 0$, there exists an $N \in \mathbb{N}$ so that, for $n \geq N$, $na_n \geq \ell/2$; equivalently, $a_n \geq (\ell/2)/n$. As the harmonic series diverges, so does $\sum_n (\ell/2)/n$ (algebraic limit theorems) and then so does $\sum_n a_n$ (order limit theorems).
(*) We will now prove that if $\sum_n x_n$ converges absolutely, and y_n is a bounded sequence, then $\sum x_n y_n$ converges absolutely. By the assumption, there exists a real number M so that $|y_n| \leq M$ for all $n \in \mathbb{N}$. Then $|x_n y_n| = |x_n||y_n| \leq M|x_n|$. As $\sum_n |x_n|$ converges, so does $\sum_n M|x_n|$ (algebraic limit theorems) and then so does $\sum_n |x_n y_n|$ (order limit theorems).
(b) Use the above with $x_n = 1/n^2$ (that converges absolutely) and $y_n = n^2 a_n$ (that is convergent, thus bounded). As $x_n y_n = a_n$, we conclude that $\sum a_n$ converges absolutely. (The book makes the assumption that $a_n > 0$, which is not necessary.)

2.7.8. (a) True. This follows from 2.7.7(*), as $a_n \to 0$ and is therefore bounded.
(b) Not true. Consider $a_n = (-1)^{n+1}/\sqrt{n}$, and $b_n = (-1)^{n+1}/\sqrt{n}$. Then $\sum_n a_n$ converges by the alternating series test, and $b_n \to 0$, but $\sum_n a_n b_n$ is the harmonic series.
(c) True. If $\sum_n n^2 a_n$ converges, then $n^2 a_n \to 0$, and then from 2.7.7(b), $\sum_n a_n$ converges absolutely.

2.7.11. For a simple example, take $(a_n) = (1, 0, 1, 0, \ldots)$ and $(b_n) = (0, 1, 0, 1, \ldots)$. For the more challenging problem, an explicit assignment for a_n and b_n is difficult; instead, we construct the sequences recursively. Let x_n be any sequence with positive decreasing terms, such that $\sum_n x_n$ converges, say $x_n = 1/n^2$.

Pick $n_1 \in \mathbb{N}$ so that $n_1 x_1 \geq 1$, and let $a_k = x_1$, $b_k = x_k$, for $k = 1, \ldots, n_1$.

Then pick $n_2 \in \mathbb{N}$ so that $(n_2 - n_1)x_{n_1+1} \geq 1$, and let $a_k = x_k$, $b_k = x_{n_1+1}$, for $k = n_1 + 1, \ldots, n_2$.

Then pick $n_3 \in \mathbb{N}$ so that $(n_3 - n_2)x_{n_2+1} \geq 1$, and let $a_k = x_{n_2+1}$, $b_k = x_k$, for $k = n_2 + 1, \ldots, n_3$.

And so on, creating long intervals on which one of the two sequences is constant while the other equals to x_k. By construction, $\min\{a_k, b_k\} = x_k$ and series $\sum_k a_k$ and $\sum_k a_k$ have unbounded partial sums.