Math 127A, Fall 2019.

HW 6 Solutions (mostly adapted from Abbott's Instructor's Manual)

3.2.1. (b) I will use G instead of O. (b) For example, $G_n = (-1-1/n, 1+1/n)$. Then $\cap_n G_n = [-1,1]$, a closed interval.

3.2.2. For the set A:
(a) The limit of even terms is 1 and the limit of odd terms is -1. Those are the two limit points. Note than $1 \in A$.
(b) Not closed. Every element of A is strictly larger than -1, so the limit point -1 is not included in the set.
Not open. No open interval is included in A. (Also follows from (c).)
(c) All points of A other than 1 are isolated.
(d) $\overline{A} = A \cup \{-1\}$.
For the set B:
(a) Every $x \in [0,1]$ is a limit of sequence of rational numbers in $(0,1)$ different from x, so the set of limit points in $[0,1]$.
(b) Not closed. Irrational numbers in $[0,1]$ are not in B but are limit points. (The same is true for the points 0 and 1.)
Not open. No open interval is included in B.
(c) No point of B is isolated, from (a).
(d) $\overline{B} = [0,1]$.

3.2.3. (a) Neither, as $\overline{\mathbb{Q}} = \mathbb{R}$, and $\mathbb{Q}$ contains no open intervals. Any irrational number is a limit point not in the set.
(b) Closed, but not open. It has no limit points, and contains no open interval.
(c) Open, as the set is equals $(-\infty, 0) \cup (0, \infty)$, a union of two open intervals, but not closed, as the complement, the singleton $\{0\}$ is not open. Also, 0 is a limit point not in the set.
(d) Neither. The infinite sum $\sum_{n=1}^{\infty} 1/n^2$ is not in the set (as it is strictly larger than any of its elements), but is in its closure. It is not open as it contains no open interval.
(e) Closed, as it has no limit points because $\sum_{k=1}^{n} 1/k \to \infty$. Not open, as it contains no open interval.

3.2.4. (a) If $s \notin \overline{A}$, then for some $\epsilon > 0$, $(s-\epsilon, s+\epsilon) \cap A = \emptyset$. As s is an upper bound of A, $(s, \infty) \cap A = \emptyset$. So we proved that $(s-\epsilon, \infty) \cap A = \emptyset$, but this means that $s - \epsilon$ is an upper bound for A, and so s is not the least upper bound for A. Contradiction.
(b) No. If $s \in A$, then for some $\epsilon > 0$, $(s-\epsilon, s+\epsilon) \subseteq A$. In particular, $s + \epsilon/2 \in A$, so s is not an upper bound for A.

3.2.7. (a) Assume x is a limit point of L, and pick $\epsilon > 0$. We need to show that there is an $a \in A \cap V_\epsilon(x)$, such that $a \neq x$. First, there is an $\ell \in L$, $\ell \neq x$, so that $|x - \ell| < \epsilon/2$. Further, as ℓ is a limit point of A, there is an $a \in A$ so that $|a - \ell| < \epsilon/2$, and $|a - \ell| < |\ell - x|/2$. Then $a \in V_\epsilon(x)$ as $|a - x| = |(a - \ell) + (\ell - x)| \leq \epsilon/2 + \epsilon/2 = \epsilon$. Also, $|a - x| \geq |\ell - x| - |a - \ell| > |\ell - x|/2 > 0$, so $a \neq x$.

3.2.11. (a) As $A \subseteq A \cup B$, $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $\overline{B} \subseteq \overline{A \cup B}$, therefore $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$. Moreover, $\overline{A} \cup \overline{B}$ is closed, as a union of two closed sets, and $A \cup B \subseteq \overline{A} \cup \overline{B}$, and $\overline{A \cup B}$ is the smallest closed set that includes $A \cup B$, so $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. The two inclusions prove equality.
(b) No. Take A_n to be closed sets, whose union is not closed, e.g., $A_n = [1/n, 1]$. Then $\cup_n A_n = (0, 1]$, $\overline{\cup_n A_n} = [0, 1]$, but $\cup_n \overline{A_n} = \cup_n A_n = (0, 1]$.