Math 127A, Fall 2019.

HW 7 Solutions (mostly adapted from Abbott's Instructor's Manual)

3.3.1. We will give a proof for $\sup K$; the proof for $\inf K$ is the same. As K is bounded, $\sup K$ is a real number. Further, by Problem 3 on Discussion 7, $\sup K \in \overline{K}$, as it is either in K or a limit point of K. As K is also closed, $\overline{K} = K$ and thus $\sup K \in K$.

3.3.2. (a) Not compact as not bounded. The sequence $a_n = n \in \mathbb{N}$ has no convergent subsequence as it diverges to ∞. (We did this in class.)
(b) Not compact as not closed. Take any sequence of rational numbers in $[0, 1]$ converging to an irrational number, say to $\sqrt{2}/2$.
(d) Not compact as not closed. Let $s_n = 1 + 1/2^2 + \ldots + 1/n^2$. Then s_n converges to a limit not in the set, and any subsequence converges to the same limit.
(e) Compact. The only limit point is 1, which is included in the set, so the set is closed and bounded.

3.3.4. (a) Compact. As a subset of the bounded set K, $K \cap F$ is bounded. As the intersection of two closed sets, $K \cap F$ is closed. By Heine-Borel Theorem, $K \cap F$ is compact.
(b) The set is closed (as closure of a set), but not necessarily compact. For example, if $F = K = \{0\}$, then $F^c = K^c = \mathbb{R} \setminus \{0\}$ and so $\overline{F^c \cup K^c} = \mathbb{R}$ which is not bounded and thus not compact.
(c) Not necessarily closed. For example, $K = [0, 1]$ and $F = \{1\}$ results in $K \setminus F = [0, 1)$, which is not closed.
(d) Compact. The set is closed as closure of a set. It is also bounded: $K \cap F^c \subseteq K$ and therefore, as K is closed, $\overline{K \cap F^c} \subset K$, and therefore, as K is bounded, $\overline{K \cap F^c}$ is bounded. Therefore, $\overline{K \cap F^c}$ is compact by Heine-Borel Theorem.

3.3.5. (a) Yes, as it is closed and bounded.
(b) No. Take $K_n = [-n, n]$. Then $\cup_{n=1}^{\infty} K_n = \mathbb{R}$, which is not compact. (c) No. Take $A = (0, 1)$, $K = [-1, 1]$; then $A \cap K = A$, which is not closed thus not compact.
(d) No. Take $F_n = [n, \infty)$. These are closed nested intervals, but $\cap_{n=1}^{\infty} F_n = \emptyset$.

3.3.11. (b) Let x_n be an increasing sequence of distinct *irrational* numbers, with $x_1 < 0$, that converges to $\sqrt{2}/2$, say $x_n = \sqrt{2}/2 - 1/n$. Then $\mathcal{C} = \{(x_1, x_2), (x_2, x_3), \ldots\} \cup \{(\sqrt{2}/2, 2)\}$ is a pairwise disjoint infinite open cover of $[0, 1] \cap \mathbb{Q}$ with no finite subcover — in fact, if we remove *any* set from $\mathcal{C}$, we no longer have a cover

3.3.13. Such sets must be finite: as singletons are closed, and every set is covered by its singletons, a clompact set must be a finite union of singletons, thus finite. On the other hand, every finite set is clopen, as any cover of a finite set admits a finite subcover.

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